# Course 424

# Group Representations I

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Walton Theatre   Friday, 10 January 2003   16:00–18:00

*Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.*

1. Define a *group representation*. What is meant by saying that 2 representations $\alpha, \beta$ are *equivalent*?

   Determine all 2-dimensional representations of $D_5$ up to equivalence, from first principles.

2. What is meant by saying that the representation $\alpha$ is *simple*?

   Determine all simple representations of $D_4$, from first principles.

3. What is meant by saying that the representation $\alpha$ is *semisimple*?

   Prove that every finite-dimensional representation $\alpha$ of a finite group over $\mathbb{C}$ is semisimple.

   Show that the natural representation of $S_n$ in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

*Continued overleaf*

4. Define the *character* $\chi_\alpha$ of a representation $\alpha$.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations $\alpha, \beta$. State and prove a formula expressing $I(\alpha, \beta)$ in terms of $\chi_\alpha, \chi_\beta$.

   Show that the simple parts of a semisimple representation are unique up to order.

5. Draw up the character table of $S_4$, explaining your reasoning throughout.

   Determine also the *representation ring* of $S_4$, ie express each product of simple representations of $S_4$ as a sum of simple representations.

6. Explain how a representation $\beta$ of a subgroup $H \subset G$ *induces* a representation $\beta^G$ of $G$.

   State (without proof) a formula for the character of $\beta^G$ in terms of that of $\beta$.

   Determine the characters of $S_4$ induced by the simple characters of $S_3$, expressing each induced character as a sum of simple parts.

7. Determine the character table of the quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}.$$

8. Show that the number of simple representations of a finite group $G$ is equal to the number of conjugacy classes in $G$.