Course 424
Group Representations

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Attempt 7 questions. (If you attempt more, only the best 7 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Find all representations of D_4 of degree 2 (up to equivalence).

   What is meant by saying that a representation α is *simple*? Find all simple representations of S_3 from first principles.

   Show that a simple representation of a finite group G necessarily has degree $\leq \|G\|$.

2. Draw up the character table for S_4.

   Determine also the representation-ring for this group, ie express the product $\alpha\beta$ of each pair of simple representation as a sum of simple representations.

3. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then

$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

4. Determine the conjugacy classes in A_5, and draw up the character table for this group.

5. Show that a simple representation of an abelian group is necessarily of degree 1.

   Determine the simple representations of $\mathrm{U}(1)$.

   List the conjugacy classes in $\mathrm{O}(2)$, and determine the simple representations of this group.

6. Determine the conjugacy classes in $\mathrm{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j+1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$).

   Express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

7. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

   Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

   Which of the 4 matrices X above are themselves expressible in the form $X = e^Y$ for some real matrix Y? Which are expressible in this form with a complex matrix Y? (Justify your answers in all cases.)

8. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{SL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{C}), \mathrm{Sp}(n)?$$

9. Determine the Lie algebras of SU(2) and SO(3), and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

10. Define the Killing form of a linear group, and determine the Killing form of SU(2) (or Sp(1)).

    Show that if the linear group G is compact then its Killing form K is negative-indefinite (or negative-definite).

    What is the condition for it to be negative-definite?