Problem Solving
Answers to Problem Set 8

11 July 2012

1. Let C be a nonempty closed bounded subset of the real line and $f : C \to C$ be a non-decreasing continuous function. Show that there exists a point $p \in C$ such that $f(p) = p$.

   (A set is closed if its complement is a union of open intervals. A function g is non-decreasing if $g(x) \leq g(y)$ for all $x \leq y$.)

   Answer: *Let*
   $$a = \inf C, \ b = \sup C,$$
   and let
   $$[a, b] \setminus C = \bigcup (c_i, c_{i+1})$$
   where the disjoint open intervals are arranged in order of decreasing size.

   Then we can extend the function $f : C \to C$ to a continuous non-decreasing function $f : [a, b] \to [a, b]$ by defining $f(x)$ linearly on (a_i, b_i) from $f(a_i)$ to $f(b_i)$ (the end-values being already defined since $a_i, b_i \in C$).

   We know that the extended function has a fixed point. This is trivial unless $f(a) > a$ and $f(b) < b$. Otherwise let
   $$c = \inf\{x \in [a, b] : f(x) \leq x\}.$$
   It is clear by continuity that $f(c) \leq c$. Also if $f(c) < c$ then again by continuity we can find $d < c$ such that $f(d) < d$. It follows that
   $$f(c) = c.$$

If $c \in C$ then all is done.

Suppose

$$c \in (a_i, b_i).$$

Since $a_i < c$ it follows from the definition of c that

$$f(a_i) > a_i.$$

(For c was the smallest point such that $f(c) \le c$.)

But

$$f(a_i) \le c,$$

since f is non-decreasing.

Since $f(a_i) \in C$, it follows that

$$f(a_i) \le a_i.$$

This is a contradiction. So c must be in C, and the result is proved.

2. *Modified* Let $P(x)$ be a polynomial with integer coefficients of the form

$$P(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0.$$

Show that if the degree n is odd, the constant term a_0 is odd, and $P(x)$ has an odd number of odd coefficients, then $P(x)$ has at least one irrational real root.

Answer: *Apologies for the absurdly incorrect problem originally set here.*

Since n is odd, there are an odd number of real roots (including any multiplicities).

By Gauss' Lemma, any rational roots are in fact integers.

Since a_0 is odd, no even integer can be a root.

Since the number of odd coefficients is odd, no odd integer can be a root.

Hence there must be an irrational real root.