Problem Solving
Answers to Problem Set 3

06 July 2012

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x, y) = (x^2 - y^2)e^{-x^2-y^2}$
   (a) Prove that f attains its minimum and maximum.
   (b) Determine all points (x, y) such that
   $$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$
   and determine for which of them f has global or local minimum or maximum.

Answer:

(a) Let
$$R = x^2 + y^2.$$
Then
$$|f(x, y)| \leq Re^{-R}.$$
Since the function on the right $\to 0$ *as* $R \to \infty$*, while*
$$f(1, 0) = e^{-1},\ f(0, 1) = -e^{-1},$$
it follows that
$$|f(x, y)| < e^{-1}$$
if $R > R_0$*, for some* $R_0 > 0$*. Hence the minimum and maximum of* f *must be the minimum and maximum inside the bounded region* $R \leq R_0$*. But we know that a continuous function will attain its minimum and maximum within a bounded region.*

(b) Since

$$f(y,x) = -f(x,y),$$

it is sufficient to find stationary points when $x \geq y$.
If $x = y$ *then* $f(x,y) = 0$. *Differentiating at* (x,x) *with respect to* x,

$$\frac{\partial f}{\partial x} = 2xe^{-2x^2}.$$

Similarly

$$\frac{\partial f}{\partial y} = -2xe^{-2x^2}.$$

Thus the only stationary point on the line $x = y$ *occurs at* $(0,0)$. *Since*

$$f(\epsilon, 0) > 0, f(0, \epsilon) < 0$$

for small ϵ, *it follows that* $(0,0)$ *is not a local minimum or maximum.*
Now let us assume $x > y$. *Taking logarithms,*

$$\log f = \log(x^2 - y^2) - x^2 - y^2.$$

Thus

$$\frac{\partial f/\partial x}{f} = \frac{2x}{x^2 - y^2} - 2x,$$

while

$$\frac{\partial f/\partial y}{f} = -\frac{2y}{x^2 - y^2} - 2y,$$

We are assuming that $(x,y) \neq (0,0)$. *If* $x = 0$ *but* $y \neq 0$ *then*

$$y^2 = 1.$$

Hence $y = -1$ *since we are assuming that* $x > y$. *Similarly, if* $y = 0$ *but* $x \neq 0$ *then*

$$x^2 = 1,$$

ie $x = 1$.
If $x \neq 0$ *and* $y \neq 0$ *then*

$$x^2 - y^2 = 1 \text{ and } x^2 - y^2 = -1,$$

which is impossible.

Thus the stationary points are:

$$(0,0), (0,\pm 1), (\pm 1, 0).$$

Since

$$f(0,\pm 1) = -e^{-1},$$

and the global minimum is attained, these must be global minima. Similarly, since

$$f(\pm 1, 0) = e^{-1},$$

these must be global maxima.

2. A sequence $u(n)$ satisfies the relation

$$u(n+2) = [u(n+1)]^2 - u(n),$$

with $u(1) = 13$ and $u(2) = 45$. Prove that 2012 divides infinitely many terms of the sequence.

Answer: *It is sufficient to consider the sequence* $\bmod 2012$.

Note that we can continue the sequence backwards as well as forwards. In particular we can go back to $u(0)$, *with*

$$u(0) = 45^2 - 13 = 2025 - 13 = 2012.$$

Now consider consecutive pairs

$$(u(n), u(n+1)) \bmod 2012.$$

Since there are only a finite number of pairs modulo 2012, *there must be a repeat. Let the first repeat be*

$$u(n+r) \equiv u(n),\ u(n+r+1) \equiv u(n+1) \pmod{2012}$$

From the recurrence relation, if $n > 0$.

$$u(n+r-1) \equiv u(n-1) \pmod{2012}$$

taking the repetition one step earlier.

Hence the first repetition comes at the beginning,

$$u(r) \equiv u(0),\ u(r+1) \equiv u(1) \pmod{2012}$$

But then it follows that the sequence must recur every r steps, and in particular

$$0 \equiv u(0) \equiv u(r) \equiv u(2r) \equiv u(3r) \equiv \cdots \pmod{2012}.$$