Maths Monthly Problems

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MM-11656 The *sign chart* of a polynomial f with real coefficients is the list of successive pairs (ϵ, σ) of signs of (f, f') on the intervals separating real zeros of ff', together with the signs at the zeros of ff' themselves, read from left to right. Thus, for $f = x^3 - 3x^2$, the sign chart is $((1,-1),(0,0),(-1,-1),(0,-1),(1,-1),(1,0),(1,1))$. As a function of n, how many distinct sign charts occur for polynomials of degree n?

MM-11657 Given a set V of n points in $\mathbb{R}^2$, no three of them collinear, let E be the set of $\binom{n}{2}$ line segments joining distinct elements of V.

1. Prove that if $n \not\equiv 2 \pmod 3$, then E can be partitioned into triples in which the length of each segment is greater than the sum of the other two.
2. Prove that if $n \equiv 2 \pmod 3$ and e is an element of E, then $E \setminus \{e\}$ can be so partitioned.

MM-11658 Let V be the vector space over $\mathbb{R}$ of all (countably) infinite sequences $(x_1, x_2, \dots)$ of real numbers, equipped with the usual addition and scalar multiplication. For $v \in V$, say that v is *binary* if $v_k \in \{0,1\}$ for $k \geq 1$, and let B be the set of all binary members of V. Prove that there exists a subset I of B with cardinality $2^{\aleph_0}$ that is linearly independent over $\mathbb{R}$. (An infinite subset of a vector space is linearly independent if all of its finite subsets are linearly independent.)

MM-11659 Let x be real with $0 < x < 1$, and consider the sequence $\langle a_n \rangle$ given by $a_0 = 0, a_1 = 1$, and, for $n > 1$,

$$a_n = \frac{a_{n-1}^2}{xa_{n-2} + (1-x)a_{n-1}}.$$

Show that

$$\lim_{n\to\infty} \frac{1}{a_n} = \sum_{k=-\infty}^{\infty} (-1)^k x^{k(3k-1)/2}.$$

MM-11660 Consider the following differential equation:

$$s''(t) = -s(t) - s(t)^2 \operatorname{sgn}(s'(t)),$$

where $\operatorname{sgn}(u)$ denotes the sign of u. Show that if $s(0) = a$ and $s'(0) = b$ with $ab \neq 0$, then (s, s') tends to $(0, 0)$ with $\sqrt{s^2 + s'^2} \leq C/t$ as $t \to \infty$, for some $C > 0$.

MM-11661 Find every function f on $\mathbb{R}^+$ that satisfies the functional equation

$$(1-z)f(x) = f\left(\frac{1-z}{z} f(xz)\right)$$

for $x > 0$ and $0 < z < 1$.

MM-11662 Let $ABCD$ be the vertices of a square, in that order. Insert P and Q on AB (in the order $APQB$) so that each of P and Q divides AB 'in extreme and mean ratio' (that is, $|AB|/|BQ| = |BQ|/|QA|$ and $|AB|/|AP| = |AP|/|PB|$). Likewise, place R and S on CD so that $CRSD$ is divided in the same proportions as $APQB$. The four intersection points of $AR, BS, CP,$ and DQ are called the *harmonious quartet* of the square on its *base pair* (AB, CD). They form a rhombus whose long diagonal has length $(\sqrt{5}+1)/2$ times the length of its short diagonal.

Given a cube, create the harmonious quartet for each of its six faces, using each edge as part of a base pair exactly once, according to this scheme: label the vertices of one face of the cube $ABCD$ and the corresponding vertices of the opposite face $A'B'C'D'$. Pair AB with CD, AA' with BB', and BC with $B'C'$. The rest of the pairings are then forced: $A'B'$ with $C'D'$, AD with $A'D'$, and CC' with DD'. This generates 24 points.

1. Show that these 24 points are a subset of the 32 vertices of a *rhombic tricontahedron* (a convex polyhedron bounded by 30 congruent rhombic faces, meeting three each across their obtuse angles at 20 vertices, and five each across their acute angles at 12 vertices), and find a construction for the remaining eight vertices.
2. Show, moreover, that the 12 end points of the longer diagonals of the constructed rhombi are the vertices of an icosahedron I, and these diagonals are edges of the icosahedron.
3. Show that the 12 end points of the shorter diagonals of the constructed rhombi, together with the eight additional vertices of the tricontahedron, are the vertices of a dodecahedron. Show also that these shorter diagonals are edges of that dodecahedron.