Maths Monthly Problems

July 2012*

MM-11649 Let p be real with $p > 1$. Let $(x_0, x_1, \ldots)$ be a sequence of non-negative real numbers. Prove that

$$\sum_{j=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{x_k}{j+k+1}\right)^p < \infty \implies \sum_{j=0}^{\infty}\left(\frac{1}{j+1}\sum_{k=0}^{j}x_k\right)^p < \infty.$$

MM-11650 Evaluate

$$\int_{x=0}^{\infty}\int_{y=x}^{\infty} e^{-(x-y)^2}\sin^2(x^2+y^2)\frac{x^2-y^2}{(x^2+y^2)^2}\,dy\,dx.$$

MM-11651 Show that the equation

$$\lfloor\frac{n+1}{\phi}\rfloor = n - \lfloor\frac{n}{\phi}\rfloor + \lfloor\frac{\lfloor n/\phi\rfloor}{\phi}\rfloor - \cdots$$

holds for every nonnegative integer n if and only if $\phi = (1+\sqrt{5})/2$.

*Solutions should be submitted by 31 October

MM-11652 For $a, b, c, d \in \mathbb{R}$, and for non-negative integers i, j and n, let

$$t_{ij} = \sum_{s=0}^{i} \binom{n-i}{j-s} \binom{i}{s} a^{n-i-j+s} b^{j-s} c^{i-s} d^{s}.$$

Let $T(a, b, c, d, n)$ be the $(n+1)$-by-$(n+1)$ matrix with (i, j)-entry given by t_{ij} for $i, j \in \{0, \ldots, n\}$. Show that

$$\det T(a, b, c, d, n) = (ad - bc)^{n(n+1)/2}.$$

MM-11663 Determine all entire functions f that satisfy, for all complex s and t, the functional equation

$$f(s+t) = \sum_{k=1}^{n-1} f^{(n-1-k)}(s) f^{(k)}(t),$$

Here, $f^{(m)}$ denotes the mth derivative of f.

MM-11664 Let Cl denote the *Clausen* function, given by $\mathrm{Cl}(\theta) = \sum_{n=1}^{\infty} \sin(n\theta)/n^2$. Let ζ denote the Riemann zeta function.

1. Show that
$$\int_{y=0}^{2\pi} \int_{y=0}^{2\pi} \log(3+2\cos x+2\cos y+2\cos(x-y)\; dx\, dy = 8\pi\, \mathrm{Cl}(\pi/3).$$
2. Show that
$$\int_{y=0}^{\pi} \int_{y=0}^{\pi} \log(3+2\cos x+2\cos y+2\cos(x-y)\; dx\, dy = \frac{28}{3}\zeta(3).$$

MM-11665 Let $ABCD$ be a convex quadrilateral, and let α, β, γ and δ be the radian measures of angles DAB, ABC, BCD and CDA, respectively. Suppose $\alpha + \beta > \pi$ and $\alpha + \delta > \pi$, and let $\eta = \alpha + \beta - \pi$ and $\phi = \alpha + \delta - \pi$. Let a, b, c, d, e, f be real numbers with $ac = bd = ef$. Show that if $abc > 0$, then

$$a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta + e\cos\eta + f\cos\phi \le \frac{be}{2a} + \frac{cf}{2b} + \frac{de}{2c} + \frac{af}{2d},$$

while for $abc < 0$ the inequality is reversed.