DUMS Intervarsity Team Selection Test

Easter 2009

Time allowed: 90 minutes

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. What are the last 3 digits of 2009^{2009}?

2. What is the first digit of 1001^{1001}?

3. Show that, in any collection of 52 distinct positive integers, there are two distinct numbers whose sum or difference is divisible by 100.

4. Six positive integers are written on the faces of a cube. At each vertex, the numbers on the 3 adjacent faces are multiplied. The sum of these 8 products is 105. What is the sum of the 6 numbers on the faces?

5. Find all integer solutions of

$$8xy + 5x + 3y = 0.$$

6. Suppose Ireland and Wales are equally strong at rugby. Which is more likely, that Ireland wins 3 games out of 4, or that Wales wins 5 games out of 8? (Ignore the possibility of draws.)

7. Given a point P and a circle Γ, suppose a line l through P cuts Γ in X, Y. Show that $PX.PY$ is independent of l.

More questions overleaf!

8. Suppose $p(x)$ is a polynomial with integer coefficients such that

$$p(0) = p(1) = 2009.$$

Show that $p(x)$ has no integer zeros.

9. Suppose the sequence x_n satisfies

$$\lim_{n \to \infty} (x_{n+1} - x_n) = 0.$$

Show that

$$\lim_{n \to \infty} \frac{x_n}{n} = 0.$$

10. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$, not identically zero, such that

$$f'(x) = f(x + 1)$$

for all x?