DUMS Intervarsity Team Selection Test

Easter 2009

Time allowed: 90 minutes

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. What are the last 3 digits of 2009^{2009}?

   Answer: *It is sufficient to determine*

$$9^{2009} \bmod 1000,$$

   since $a \equiv b \mod n \implies a^n \equiv b^n \mod n$.

   By the binomial theorem,

$$\begin{aligned} 9^{2009} &= (10-1)^{2009} \\ &\equiv (-1)^{2009} + 2009 \cdot 10 \cdot (-1)^{2008} + \binom{2009}{2} 10^2 (-1)^{2007} \bmod 1000 \\ &\equiv -1 + 90 - \frac{9 \cdot 8}{2} 100 \bmod 1000 \\ &\equiv -1 + 90 - 600 \bmod 1000 \\ &\equiv 489 \bmod 1000, \end{aligned}$$

   ie the last 3 digits are 489.

2. What is the first digit of 1001^{1001}?

   Answer: *We know that*

$$(1 + \frac{1}{n})^n \to e$$

More questions overleaf!

as $n \to \infty$.

The first digit of 1001^{1001} *is the same as the first digit of* $(1 + \frac{1}{1000})^{1001}$.

But

$$(1 + \frac{1}{1000})^{1000} \approx e = 2.6\ldots.$$

The additional factor $(1 + 1/1000)$ *will not affect the first digit. Thus the first digit is 2.*

[And the first 2 digits are 26.

If asked to justify the argument above we would go back to the proof that $(1 + 1/n)^n \to e$, *by taking logarithms, with*

$$\log(1 + 1/n) = 1/n - 1/2n^2 + 1/3n^3 - \cdots,$$

so that

$$\log(1 + 1/n)^n = 1 + O(1/n).$$

It would be a straightforward matter to verify that the error involved in omitting the terms after the first is not sufficient to alter the first, or even the first 2, digits.]

3. Show that, in any collection of 52 distinct positive integers, there are two distinct numbers whose sum or difference is divisible by 100.

   Answer: *If 0 and/or 50 are among the 52 integers, ignore them. Let the remaining integers be* $x_1, \ldots, x_{50}$. *Consider the 101 integers*

   $$\{50, \pm x_1, \ldots, \pm x_{50}\}.$$

   There are 100 residue-classes mod 100. *Hence (by the pigeon-hole principle) two of these must lie in the same residue-class, ie*

   $$x_i \equiv x_j \bmod 100 \text{ or } x_i \equiv -x_j \mod 100.$$

   In the first case $x_i - x_j$ *is divisible by 100; in the seond case* $x_i + x_j$ *is divisible by 100.*

4. Six positive integers are written on the faces of a cube. At each vertex, the numbers on the 3 adjacent faces are multiplied. The sum of these 8 products is 105. What is the sum of the 6 numbers on the faces?

   Answer: *Choose two opposite faces. Let the numbers on these be* a, b. *Let the numbers on the remaining 4 faces (going round the cube) be* $cdef$. *Then the sum of the products involving the first face is*

   $$a(cd + de + ef + fc) = a(c + e)(d + f).$$

Similarly, the sum of the products involving the second face is

$$b(cd + de + ef + fc) = a(c + e)(d + f).$$

Hence the sum of all the products is

$$(a + b)(c + e)(d + f).$$

But

$$105 = 3 \cdot 5 \cdot 7.$$

Hence

$$a + b, c + e, d + f = 3, 5, 7$$

in some order. We conclude that

$$a + b + c + d + e + f = 3 + 5 + 7 = 15.$$

5. Find all integer solutions of

$$8xy + 5x + 3y = 0.$$

Answer: *[There are various approaches to this very simple problem.] Multiplying the equation by 8,*

$$64xy + 40x + 24y = 0,$$

ie

$$(8x + 3)(8y + 5) = 15.$$

Thus

$$(8x+3, 8y+5) = (1, 15), (-1, -15), (15, 1), (-15, -1), (3, 5), (-3, -5), (5, 3), (-5, -3).$$

Each of these determines x, y. The only cases giving integer values to x, y are

$$(8x + 3, 8y + 5) = (3, 5) \implies (x, y) = (0, 0)$$
$$(8x + 3, 8y + 5) = (-5, -3) \implies (x, y) = (-1, -1).$$

6. Suppose Ireland and Wales are equally strong at rugby. Which is more likely, that Ireland wins 3 games out of 4, or that Wales wins 5 games out of 8? (Ignore the possibility of draws.)

   Answer: *The probability of Ireland winning 3 games out of 4 is*

$$\binom{4}{3}(\frac{1}{2})^4 = \frac{1}{4}.$$

   The probability of Wales winning 5 games out of 8 is

$$\binom{8}{5}(\frac{1}{2})^8 = \frac{7}{32}.$$

   Thus it is more likely that Ireland wins 3 games out of 4.

7. Given a point P and a circle Γ, suppose a line l through P cuts Γ in X, Y. Show that $PX.PY$ is independent of l.

   Answer: *Suppose $PXY, PX'Y'$ are two such lines. Consider the triangles PXY', PYX'.*

   Recall that the angle AXB subtended by a chord AB in a circle is constant. Hence the angles $\angle PXY' = \angle X'XY$ and $\angle PYX' = \angle Y'YX$ are equal.

   The two triangles also have the angle $\angle XPX'$ in common. Hence the two triangles have the same angles, and so are similar.

   It follows that

$$\frac{PX}{PY'} = \frac{PY}{PX'}$$

   ie

$$PX \cdot PX' = PY \cdot PY'.$$

8. Suppose $p(x)$ is a polynomial with integer coefficients such that

$$p(0) = p(1) = 2009.$$

   Show that $p(x)$ has no integer zeros.

   Answer: *Consider the polynomial*

$$f(x) = p(x) - 2009.$$

We have

$$f(0) = f(1) = 0.$$

Hence

$$f(x) = x(x-1)q(x)$$

for some polynomial $q(x)$*, ie*

$$p(x) = x(x-1)q(x) + 2009.$$

If

$$p(n) = 0$$

then

$$n(n-1) \mid 2009.$$

But

$$2009 = 7 \cdot 287.$$

Thus

$$n \in \{\pm 1, \pm 7, \pm 287, \pm 2009\}.$$

But in none of these cases does $(n-1) \mid 2009$.

9. Suppose the sequence x_n satisfies

$$\lim_{n\to\infty} (x_{n+1} - x_n) = 0.$$

Show that

$$\lim_{n\to\infty} \frac{x_n}{n} = 0.$$

Answer: *Given* $\epsilon > 0$ *we can find* N *such that*

$$|x_{n+1} - x_n| < \epsilon/2 \text{ if } n \geq N.$$

If $x \geq N$ *then*

$$x_n = (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{N+1} - x_N) + x_N.$$

Hence

$$\begin{aligned} |x_n| &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{N+1} - x_N| + |x_N| \\ &\leq (n-N)\epsilon/2 + |x_N| \end{aligned}$$

Thus

$$|a_n/n| \leq \epsilon/2 + |x_N/n|.$$

Now choose M such that

$$|x_N/n| < \epsilon/2$$

if $n \geq M$. Then

$$|x_n/n| < \epsilon$$

if $n \geq \max(M, N)$.

Hence

$$x_n/n \to 0.$$

10. Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$, not identically zero, such that

$$f'(x) = f(x+1)$$

for all x?

Answer: *Let us try*

$$f(x) = e^{cx}.$$

This will satisfy the given equation if

$$ce^{cx} = e^{c(x+1)} = e^c e^{cx},$$

ie if

$$c = e^c.$$

Now it is easy to see that $c \neq e^c$ for any real number c; for if $c < 0$ then $e^c > 0$, while if $c > 0$ then $e^c > c$.

However, we may be able to find a complex number c such that

$$e^c = c;$$

and then

$$f(x) = \Re(e^{cx}) = \cos(cx)$$

will satisfy the given relation.

Thus there is a solution provided the function

$$f(z) = e^z - z$$

has a zero.