Irish Intervarsity Mathematics Competition 1994

University College Dublin

1. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}.$$

   Answer: *These questions can nearly always be solved by using partial fractions, if they have a solution. But sometimes a little subtlety shortens the calculation.*

   In this case
$$n^4+4 = (n-\sqrt{2}\omega)(n-\sqrt{2}\omega^3)(n-\sqrt{2}\omega^5)(n-\sqrt{2}\omega^7)$$
   where $\omega = e^{2\pi/8}$.

   We can combine conjugate factors to give 2 real quadratics
$$x^4+4 = (x^2+ax+b)(x^2+cx+d).$$
   From the coefficients of x^3 *and* x, *we must have* $c=-a,\ b=d$.
$$x^4+4 = (x^2+ax+b)(x^2-ax+b).$$

Thus

$$-a^2 + 2b = 0, \; b^2 = 4.$$

Hence $b = 2, \; a = 2$, *yielding*

$$x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2).$$

Now

$$\frac{1}{n^2 - 2n + 2} - \frac{1}{n^2 + 2n + 2} = \frac{4n}{n^4 + 4}$$

Hopefully, the 2 terms will cancel out in some way. In fact

$$(n + 2)^2 - 2(n + 2) + 2 = n^2 + 2n + 2.$$

Thus the first term with $n + 2$ *cancels out the second term with* n. *We are left with the first term for* $n = 1, 2$.

$$\sum \frac{n}{n^4 + 4} = \frac{1}{4}\left(\frac{1}{1} + \frac{1}{2}\right) = \frac{3}{8}.$$

2. Prove that there exist infinitely many positive integers n such that $2n + 1$ and $3n + 1$ are both perfect squares.

   Answer: *Suppose*

$$2n + 1 = x^2, \; 3n + 1 = y^2.$$

Then

$$3x^2 - 2y^2 = 1.$$

Conversely, if x, y *satisfy this equation then* x *is odd and*

$$n = \frac{x^2 - 1}{2}$$

satisfies the original problem. Thus the question reduces to the solution of the above diophantine equation.

This is closely related to Pell's equation

$$u^2 - 6v^2 = 1,$$

which is solved by considering

$$z = u + v\sqrt{6}.$$

For if we set

$$\tilde{z} = u - v\sqrt{6},$$

then Pell's equation can be written

$$z\tilde{z} = 1.$$

Now suppose we have one solution u, v. Then we get an infinity of solutions by considering

$$z^n = (u + v\sqrt{6})^n = U + V\sqrt{6}.$$

For then

$$\tilde{z}^n = U - V\sqrt{6},$$

and so

$$U^2 - 6V^2 = (z\tilde{z})^n = 1.$$

Going back to our equation for x, y, we see that we can write this

$$(3x)^2 - 6y^2 = 3,$$

say

$$X^2 - 6Y^2 = 3,$$

In other words,

$$(X + Y\sqrt{6})(X - Y\sqrt{6}) = 3.$$

Now suppose we have one solution (eg $X = 3,\ y = 1$). Then we can get an infinity of solutions by taking

$$z = (X + Y\sqrt{6})(U + V\sqrt{6})$$

where $U^2 - 6V^2 = 1$.

(This is the classical number theory of a real quadratic number field.)

3. Let A, C be points in the plane and B the midpoint of $[AC]$. Let S be the circle with centre A and radius $|AB|$ and T the circle with centre C and radius $|AC|$. Suppose S and T intersect in R, R'. Let S', T' be the circles with centres R, R' and radii $AR = AR'$. Suppose S' and T' intersect in A and C'. Prove that C' is the midpoint of $[AB]$.

   Answer: *Let us choose euclidean coordinates with*

$$A = (0,0),\ B = (1,0),\ C = (2,0).$$

Then S has equation

$$x^2 + y^2 - 1 = 0,$$

and T has equation

$$(x-2)^2 + y^2 - 4 \equiv x^2 - 4x + y^2 = 0.$$

The equation of the line RR' is obtained by subtracting the equations of S and T:

$$4x - 1 = 0.$$

This meets the line AB at the point

$$D = (1/4, 0).$$

Evidently RR' is the perpendicular bisector of AC'. It follows that

$$C' = (1/2, 0),$$

ie C' is the midpoint of $[AC]$.

4. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients a_i. Suppose that z is an integer and that

$$P(P(P(P(z)))) = z.$$

Prove that $P(P(z)) = z$.

Answer: *Let*

$$z_0 = z, z_1 = f(z),\ z_2 = f(z_1), z_3 = f(z_2), z_4 = f(z_3).$$

We have to show that

$$z_4 = z_0 \Longrightarrow z_2 = z_0.$$

We may suppose therefore that $z_2 \neq z_0$. This implies that $z_1 \neq z_0$.

Any change of coordinates $x \mapsto x - n$ (where $n \in \mathbb{Z}$) will replace $P(x)$ by another polynomial $P(x+n) - n$ with integral coefficients. Thus we may suppose that $z = 0$.

Then

$$z_1 = a_0.$$

It is easy to see that

$$a_0 \mid z_1, z_2, z_3.$$

On the other hand

$$0 = f(z_3) = a_0 + z_3(a_1 + a_2 z_3 + \cdots)$$

implies that

$$z_3 \mid a_0.$$

It follows that

$$z_3 = \pm a_0.$$

Since $z_3 = a_0 = z_1 \Longrightarrow z_4 = z_2 \neq 0$, *we must have*

$$z_3 = -a_0.$$

Thus our cycle is

$$0 \mapsto a_0 \mapsto z_2 \mapsto -a_0 \mapsto 0.$$

We have shown therefore that

$$z_3 + z_1 = 0 = 2z_0.$$

We have written the equation in this form so that it will remain true under any change of coordinate $x \mapsto x - n$. *Thus if we start our cycle with* z_2 *we deduce that*

$$z_1 + z_3 = 2z_1 \Longrightarrow z_1 = 0.$$

[This last trick is not essential. The result can be proved by pursuing the previous argument, showing that $z_2 = \pm 2a_0$, *and deducing that* $a_0 = \pm 1,\ \pm 2$ *or* ± 4.*]*

5. Let a, b, c $(a < b < c)$ be the lengths of the sides of a triangle opposite the interior angles A, B and C, respectively. Prove that if a^2, b^2, c^2 are in arithmetic progression, then so are $\cot(A), \cot(B), \cot(C)$.

   Answer: *We have to show that*

$$a^2 + c^2 = 2b^2 \Longrightarrow \cot A + \cot C = 2\cot B.$$

By the 'sine law' for the triangle,

$$a : b : c = \sin A : \sin B : \sin C.$$

Thus $a^2 + c^2 = 2b^2$ *if and only if*

$$\begin{aligned}
\sin^2 A + \sin^2 C &= 2\sin^2 B \\
&= 2\sin^2(A + C) \\
&= 2\sin^2 A\cos^2 C + 2\cos^2 A\sin^2 C + 4\sin A\sin C\cos A\cos C \\
&= 2\sin^2 A + 2\sin^2 C - 4\sin^2 A\sin^2 C + 4\sin A\sin C\cos A\cos C,
\end{aligned}$$

that is, if and only if

$$\sin^2 A + \sin^2 C = 4\sin^2 A\sin^2 C + 4\sin A\sin C\cos A\cos C$$

On the other hand $\cot A + \cot C = 2\cot B$ *if and only if*

$$\cot A + \cot C = -2\cot(A + C),$$

that is,

$$\frac{\cos A}{\sin A} + \frac{\cos C}{\sin C} = -2\frac{\cos(A + C)}{\sin(A + C)},$$

which simplifies to

$$\sin^2(A + C) = -2\sin A\sin C(\cos A\cos C - \sin A\sin C).$$

Expanding,

$$\begin{aligned}
&\sin^2 A\cos^2 C + \sin^2 C\cos^2 A + 2\sin A\sin C\cos A\cos C = \\
&\quad -2\sin A\sin C(\cos A\cos C - \sin A\sin C),
\end{aligned}$$

which reduces to the same condition as before:

$$\sin^2 A + \sin^2 C = 4\sin^2 A\sin^2 C + 4\sin A\sin C\cos A\cos C.$$

6. Let

$$A = \begin{pmatrix} 1994 & 1993 \\ 1995 & 1994 \end{pmatrix}.$$

Prove that A can be written as the product $X_1 X_2 \cdots X_r$ where $r \geq 1$ and each

$$X_i \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \quad (i = 1, 2, \ldots, r)$$

in exactly one way.

Answer: *The matrix A is* unimodular, *ie* $\det A = 1$. *It is well-known that the unimodular group*

$$\mathbf{SL}(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : ad - bc = 1\}$$

is generated by the matrices

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \; V = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

One can show this by multiplying successively on the right by $U^{\pm 1}$, $V^{\pm 1}$.

In our case, suppose

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (ad - bc = 1)$$

with $a, b, c, d > 0$. Consider

$$XU^{-1} = \begin{pmatrix} a & b-a \\ c & d-c \end{pmatrix} \text{ and } XV^{-1} = \begin{pmatrix} a-b & b \\ c-d & d \end{pmatrix}.$$

If $b \geq a$ then XU^{-1} has non-negative entries and is 'smaller' than X, as measured say by the sum of the entries $a + b + c + d$. If $b < a$ then XV^{-1} has non-negative entries and is 'smaller' than X. Thus by successively multiplying by U^{-1} or V^{-1} (ensuring at each stage that the entries are non-negative) we must finally arrive at the identity matrix I. In other words,

$$AX_1^{-1}X_2^{-1} \cdots X_r^{-1} = I,$$

where each X_i is either U or V. Thus

$$A = X_1 X_2 \cdots X_r.$$

7. Noah had 8 species of animals to fit into 4 cages of the ark. He planned to put two species in each cage. It turned out that for each species, there were at most three other species with which it could not share a cage. Could Noah have carried out his plan while arranging that each species shares with a compatible species?

   Answer: *There is probably a 'smarter' way of solving this than the following 'proof by cases'.*

There are 28 possible pairings, of which 16 are compatible.

Take any species X. Let the species with which X is compatible form the set

$$S = \{A, B, C, D\};$$

and let the remaining species from the set

$$T = \{a, b, c\}.$$

Consider the possible pairings inside T. There are 4 cases, according as there are 0, 1, 2 or 3 pairings.

If there are no pairings inside T, then each of a, b, c must be compatible with all of $S = \{A, B, C, D\}$. Thus we can pair off $(a, A), (b, B), (c, C), (X, D)$.

Suppose there is one pairing inside T, say (a, b). Then c is compatible with all of S while a and b are each compatible with 3 of S. Thus we know of $4 + 3 + 3 + 4$ pairings; so there must be 2 pairings within S. If (A, B) is one of these pairings, thus we can pair off $(A, B), (c, C), (X, D), (a, b)$.

Suppose there are two pairings inside T, say $(a, b), (b, c)$. Then a and b are each compatible with 3 of S, while c is compatible with 2. Thus we know of $4+3+3+2$ pairings; so there must be 4 pairings within S. If these contain a triangle they can be taken in the form $(A, B), (B, C), (C, A), (A, D)$. If not, they can be taken as $(A, B), (B, C), (C, D), (A, B)$. In either case, D must be compatible with a or b. We may suppose it is compatible with a; and then we can pair off $(A, B), (X, C), (a, D), (b, c)$.

Finally, suppose there are three pairings inside T. Then a, b, c are each compatible with 2 of S. Thus we know of $4 + 2 + 2 + 3$ pairings; so there must be 5 pairings within S. There are 6 possible pairings in S, so there is 1 'non-pairing', say (A, B). Then B must be paired with one element of T, say a; and we can pair off $(X, A), (B, a), (C, D), (b, c)$.

8. The function

$$\sum_{n=1}^{\infty} \frac{nx^n}{1 - x^n} \quad (|x| < 1)$$

is expanded as a power-series $\sum_{k=1}^{\infty} a_k x^k$. Prove that $n = 1994$ is the largest even integer with $a_n = n + 1000$.

Answer: *This is almost trivial if I have the correct question.*

The term

$$\frac{nx^n}{1 - x^n} = nx^n + nx^{2n} + nx^{3n} + \cdots)$$

will contribute n to each a_k with $n \mid k$. In other words, a_k is just the sum of the factors of *k:*

$$a_k = S(k).$$

But if $k = 2m$ then certainly k has factors $1, 2, m, k$. Thus

$$S(k) \geq k + 3 + m.$$

So if $k > 1994$, ie $m > 997$ then

$$S(k) > k + 1000.$$

Hence $k = 1994$ is the largest number with $S(k) = k + 1000$.

9. Let a, b, c be positive real numbers. Prove that

$$[(a+b)(b+c)(c+a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab+bc+ca)^{1/2}.$$

Answer: *This is a straightforward exercise in partial differentiation. Let*

$$f(a,b,c) = \frac{[(a+b)(b+c)(c+a)]^{1/3}}{(ab+bc+ca)^{1/2}}.$$

We want to determine the minimum of $f(a,b,c)$. At such a minimum,

$$\frac{\partial f}{\partial a} = \frac{\partial f}{\partial b} = \frac{\partial f}{\partial c} = 0.$$

But on differentiating $\log f$*,*

$$\frac{1}{f}\frac{\partial f}{\partial a} = \frac{(b+c)(2a+b+c)}{3(a+b)(b+c)(c+a)} - \frac{b+c}{2(ab+bc+ca)}.$$

Thus

$$2a+b+c = \frac{3(a+b)(b+c)(c+a)}{2(ab+bc+ca)}.$$

Similarly

$$a+2b+c = a+b+2c = \frac{3(a+b)(b+c)(c+a)}{2(ab+bc+ca)}.$$

Thus

$$2a+b+c = a+2b+c = a+b+2c,$$

from which we deduce that

$$a = b = c.$$

Since

$$f(a, a, a) = \frac{2}{\sqrt{3}},$$

the result follows.

10. Let $n > 1$ be a positive integer. Prove that

$$\sum_{j=1}^{n-1} j \operatorname{cosec}^2\left(\frac{\pi j}{n}\right) = \frac{n(n^2-1)}{6}.$$

Answer: *The factor j is somewhat misleading, since*

$$\operatorname{cosec}\left(\frac{\pi(n-j)}{n}\right) = \operatorname{cosec}\left(\frac{\pi j)}{n}\right).$$

Thus the jth and $(n-j)$th terms combine to give

$$n \operatorname{cosec}^2\left(\frac{\pi j)}{n}\right).$$

Hence

$$\begin{aligned} S &= \sum_{j=1}^{n-1} j \operatorname{cosec}^2\left(\frac{\pi j}{n}\right) \\ &= \frac{n}{2}\sum_{j=1}^{n-1} \operatorname{cosec}^2\left(\frac{\pi j}{n}\right) \\ &= \frac{n}{2}S', \end{aligned}$$

say. Now we are on more familiar ground. Sums like S' can usually be calculated in the following way.

Consider the equation

$$\tan n\theta = 0.$$

We can express $\tan n\theta$ as a rational function in $\cot\theta$:

$$\tan n\theta = \frac{P(\cot\theta)}{Q(\cot\theta)},$$

where $P(x), Q(x)$ are polynomials. But

$$\tan n\theta = 0 \iff n\theta = j\pi$$

for some integer j. It follows that the polynomial $P(x)$ vanishes when

$$x = \cot \frac{j\pi}{n} = \gamma_j,$$

say, for $j = 1, \ldots, n-1$. These roots are distinct; so if $P(x)$ has degree $n-1$ we must have

$$P(x) = c(x - \gamma_1) \cdots (x - \gamma_{n-1}).$$

Thus we should be able to calculate symmetric functions of the γ's from the coefficients of $P(x)$.

Let

$$u = \cot\theta,\ z = e^{i\theta}.$$

Then

$$u = i\frac{z + z^{-1}}{z - z^{-1}} = i\frac{z^2 - 1}{z^2 + 1};$$

and so

$$z^2 = \frac{u - i}{u + i}.$$

But

$$\begin{aligned} \tan n\theta &= \frac{1}{i}\frac{z^n - z^{-n}}{z^n - z^{-n}} \\ &= \frac{1}{i}\frac{z^{2n} - 1}{z^{2n} + 1} \\ &= \frac{1}{i}\frac{u - i))^n - (u + i)^n}{(u - i)^n + (u + i)^n}. \end{aligned}$$

It follows that (up to a scalar multiple)

$$\begin{aligned} P(x) &= \frac{1}{i}\left((u - i)^n - (u + i)^n\right) \\ &= -2nu^{n-1} + 2\frac{n(n-1)(n-2)}{6}u^{n-3} + \cdots . \end{aligned}$$

Thus

$$\sum \gamma_j = 0,\ \sum \gamma_j\gamma_k = -\frac{(n-1)(n-2)}{6}.$$

Now

$$\operatorname{cosec}^2\left(\frac{j\pi}{n}\right) = 1 + \gamma_j^2.$$

Thus

$$\begin{aligned}
S &= \frac{n}{2}\left(n - 1 + \sum \gamma_j^2\right) \\
&= \frac{n}{2}\left(n - 1 + (\sum \gamma_j)^2 - 2\sum \gamma_j\gamma_k\right) \\
&= \frac{n}{2}\left(n - 1 + \frac{n-1)(n-2)}{3}\right) \\
&= \frac{n}{6}\left(3n - 3 + (n-1)(n-2)\right) \\
&= \frac{n(n^2-1)}{6}.
\end{aligned}$$