



# Royal Life Ireland

## Irish Intervarsity Mathematics Competition

Trinity College Dublin 1991

9.30–12.30 February 9

*Answer all questions.  
Calculators permitted.*

1. The prime factorisations of  $r + 1$  positive integers ( $r \geq 1$ ) together involve only  $r$  primes. Prove that there is a subset of these integers whose product is a perfect square.
2. Consider a polynomial  $p(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$  which has real roots  $r_1, r_2, \dots, r_n$ . If
$$r_1^{16} + r_2^{16} + \cdots + r_n^{16} = n,$$
find all the roots.
3. An  $n$ -inch cube ( $n$  a positive integer) is painted on all sides and then cut into 1-inch cubes. If the number of small cubes with one painted side is the same as the number with two painted sides, what could  $n$  have been?
4. How many ways are there of painting the 6 faces of a cube in 6 different colours, if two colourings are considered the same when one can be obtained from the other by rotating the cube?
5. How many positive integers  $x \leq 1991$  are such that 7 divides  $2^x - x^2$ ?
6. How many ways can 1,000,000 be expressed as a product of 3 positive integers? Factorisations different only in order are considered to be the same.

7. Prove that  $2^n$  can begin with any sequence of digits.
8. Imagine a point  $P$  inside a square  $ABCD$ . If  $|PA| = 5$ ,  $|PB| = 3$  and  $|PC| = 7$ , what is the side of the square?
9. Let  $f(x)$  be a function such that  $f(1) = 1$  and, for  $x \geq 1$

$$f'(x) = \frac{1}{x^2 + f^2(x)}.$$

Prove that

$$\lim_{x \rightarrow \infty} f(x)$$

exists and is less than  $1 + \pi/4$ .

10. Prove that the number of odd binomial coefficients in each row of Pascal's triangle is a power of 2. [In Pascal's triangle

$$\begin{array}{cccc}
 & & 1 & \\
 & & 1 & 1 \\
 & 1 & 2 & 1 \\
 1 & 3 & 3 & 1 \\
 & & \vdots & 
 \end{array}$$

each entry is the sum of the entries directly above it.]