Irish Intervarsity Mathematics Competition

University College Cork 1990

1. If x, y and z are real numbers solve the equations

$$\begin{aligned} x + y + z &= \sqrt[3]{2} \\ x^2 + y^2 + z^2 &= \sqrt[3]{4} \\ x^3 + y^3 + z^3 &= 2 \end{aligned}$$

Answer: *We can simplify the equations by setting*

$$x = \sqrt[3]{2}X,\ y = \sqrt[3]{2}Y,\ z = \sqrt[3]{2}Z.$$

The equations become

$$\begin{aligned} X + Y + Z &= 1 \\ X^2 + Y^2 + Z^2 &= 1 \\ X^3 + Y^3 + Z^3 &= 1 \end{aligned}$$

From the power-sums on the left-hand side of these equations we can compute the symmetric products

$$c_1 = X + Y + Z,\ c_2 = YZ + ZX + XY,\ c_3 = XYZ.$$

The solutions to the equations are then given by

$$\{X, Y, Z\} = \{\alpha, \beta, \gamma\},$$

where α, β, γ are the roots of a cubic

$$t^3 - c_1 t^2 + c_2 t^1 - c_3 = 0.$$

But it is clear from the equations that one solution is

$$(X, Y, Z) = (1, 0, 0).$$

It follows that there are just 3 solutions:

$$(X, Y, Z) = (1, 0, 0),\ (0, 1, 0),\ (0, 0, 1),$$

2. A hand of 13 cards is dealt from a pack of 52 cards containing 13 spades, 13 diamonds, 13 clubs and 13 hearts. If the hand has 3 clubs, 4 spades, 2 diamonds and 4 hearts, it is siad to have the distribution type $4-4-3-2$, while if it has 6 hearts and 7 spades it is said to have distribution type $7-6$. How many distribution types are there? List them all.

   Answer: *We are asked, in effect, for the number $p_4(13)$ of partitions of 13 into 4 or fewer parts. It is well-known that there is no simple formula for $p_r(n)$, although there are various recurrence formulae, expressing $p_r(n)$ in terms of partitions of smaller numbers.*

   However, we are asked to list the partitions, so there seems little point in looking for a nice method. The partitions are:

$$\begin{aligned}
&13,\\
&12-1,\\
&11-2, 11-1-1,\\
&10-3, 10-2-1, 10-1-1-1,\\
&9-4, 9-3-1, 9-2-2, 9-2-1-1,\\
&8-5, 8-4-1, 8-3-2, 8-3-1-1, 8-2-2-1,\\
&7-6, 7-5-1, 7-4-2, 7-4-1-1, 7-3-3, 7-3-2-1, 7-2-2-2,\\
&6-6-1, 6-5-2, 6-5-1-1, 6-4-3, 6-4-2-1, 6-3-3-1, 6-3-2-2,\\
&5-5-3, 5-5-2-1, 5-4-4, 5-4-3-1, 5-4-2-2, 5-3-3-2,\\
&4-4-4-1, 4-4-3-2, 4-3-3-3.
\end{aligned}$$

   Thus

$$p_4(13) = 39.$$

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

Answer: *We have*

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x^n}{3^n} &= \left(1 - \frac{x}{3}\right)^{-1} \\ &= \frac{3}{3-x}.\end{aligned}$$

Differentiating,

$$\sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{3}{(3-x)^2}$$

Multiplying by x,

$$\sum_{n=1}^{\infty} \frac{nx^n}{3^n} = \frac{3x}{(3-x)^2}$$

Differentiating again,

$$\sum_{n=1}^{\infty} \frac{n^2x^{n-1}}{3^n} = \frac{3}{(3-x)^2} + \frac{6x}{(3-x)^3}.$$

Multiplying by x,

$$\sum_{n=1}^{\infty} \frac{n^2x^n}{3^n} = \frac{3x}{(3-x)^2} + \frac{6x^2}{(3-x)^3}.$$

Differentiating a third time,

$$\sum_{n=1}^{\infty} \frac{n^3x^{n-1}}{3^n} = \frac{3}{(3-x)^2} + \frac{18x}{(3-x)^3} + \frac{18x^2}{(3-x)^4}.$$

Subsituting $x = 1$,

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{n^3}{3^n} &= \frac{3}{4} + \frac{18}{8} + \frac{18}{16} \\ &= \frac{33}{8}.\end{aligned}$$

4. Find the minimum value of the real function

$$f(x) = x^x, \qquad \text{for } x > 0.$$

Answer: *We have*

$$\log f(x) = x \log x.$$

Hence

$$\frac{f'(x)}{f(x)} = \log x + 1.$$

Thus

$$f'(x) = 0 \iff \log x = -1 \iff x = e^{-1}.$$

As $x \to 0+$, $\log f(x) \to 0$ *and so* $f(x) \to 1$. *As* $x \to \infty$, $\log f(x) \to \infty$ *and so* $f(x) \to \infty$. *Thus the minimum value is*

$$f(e^{-1}) = e^{-e^{-1}} = \frac{1}{e^{1/e}}.$$

5. Prove that

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$$

Answer: *We have*

$$\tan 7\theta = \frac{P(\tan\theta)}{Q(\tan\theta)},$$

where $P(t)$ *and* $Q(t)$ *are polynomials of degree* ≤ 7 *in* t. *Now*

$$\tan 7\theta = 0 \iff \theta = \frac{n\pi}{7}$$

for some integer n. *It follows that the roots of* $P(t) = 0$ *are the 7 numbers*

$$\tan\frac{n\pi}{7}, \qquad (n = 0, 1, \ldots, 6).$$

In fact, since

$$\tan\frac{(7-n)\pi}{7} = -\tan\frac{n\pi}{7},$$

the roots are

$$0, \ \pm\tan\frac{\pi}{7}, \ \pm\tan\frac{2\pi}{7}, \ \pm\tan\frac{3\pi}{7}.$$

Hence

$$P(t) = x(at^6 + bt^4 + ct^2 + d).$$

Thus

$$\tan^2\left(\frac{\pi}{7}\right), \ \tan^2\left(\frac{2\pi}{7}\right), \ \tan^2\left(\frac{3\pi}{7}\right)$$

are the roots of

$$ax^3 + bx^2 + cx + d.$$

In particular

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = -\frac{b}{a}.$$

It only remains to compute the polynomial $P(t)$*. Writing* $\tan\theta = t$*, we have*

$$\begin{aligned}
\tan 2\theta &= \frac{2t}{1-t^2}, \\
\tan 3\theta &= \frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} \\
&= \frac{3t - t^3}{1 - 3t^2}, \\
\tan 4\theta &= \frac{2\frac{2t}{1-t^2}}{1 - \frac{4t^2}{(1-t^2)^2}} \\
&= \frac{4t(1-t^2)}{1 - 6t^2 + t^4}, \\
\tan 7\theta &= \frac{\frac{3t-t^3}{1-3t^2} + \frac{4t(1-t^2)}{1-6t^2+t^4}}{1 - \frac{3t-t^3}{1-3t^2}\frac{4t(1-t^2)}{1-6t^2+t^4}} \\
&= \frac{P(t)}{Q(t)},
\end{aligned}$$

where

$$\begin{aligned}
P(t) &= (3t - t^3)(1 - 6t^2 + t^4) + 4t(1 - t^2)(1 - 3t^2) \\
&= -t\left[t^6 - 21t^4 + 34t^2 - 7\right].
\end{aligned}$$

Hence

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$$

6. A ladder of constant length $2L$ extends from a horizontal floor to a wall which is inclined at $85°$ to the horizontal. If the ladder slips while remaining in contact with both the floor and the wall, find the locus of the mid-point of the ladder.

Answer: *Taking cartesian coordinates centred on the point where the floor and wall meet, the coordinate of the point where the ladder meets the wall is*

$$(Y \sin 5°, Y \cos 5°)$$

for some Y. *If the point where the ladder meets the floor is* $(X, 0)$ *then*

$$(X - \sin 5° Y)^2 + (\cos 5° Y)^2 = 4L^2.$$

In other words

$$X^2 + Y^2 - 2 \sin 5° XY = 4L^2.$$

The mid-point of the ladder has coordinates

$$(x, y) = \left(\frac{X + \sin 5° Y}{2}, \frac{\cos 5° Y}{2} \right).$$

Thus

$$Y = 2 \sec 5° y, \ X = 2x - 2 \tan 5° y.$$

Thus the point (x, y) *lies on the locus*

$$(x - \tan 5° y)^2 + \sec^2 5° y^2 - 2 \sin 5° (x - \tan 5° y) \sec 5° y = L^2.$$

Thus the locus of the mid-point of the ladder is the ellipse

$$x^2 + (\tan^2 5° + \sec^2 5°) y^2 - 2 \tan 5° xy = L^2;$$

or rather, that part of the ellipse between the wall and the floor.

7. Prove that 3, 5 and 7 are the only 3 consecutive odd numbers all of which are prime.

   Answer: *One of the 3 numbers* $n, n+2, n+4$ *is divisible by 3. So if all 3 are prime, one of the numbers must be 3. Thus the 3 numbers must be* $3, 5, 7$.

8. If x, y and z are positive integers, all greater than 1, find *any* solution of the equation

   $$x^x y^y = z^z.$$

   Answer: *[Solution by James Ward, of UCG] Any solution must have*

   $$z < x + y,$$

   since

   $$(x + y)^{x+y} > x^x y^y$$

(as one of the terms in the binomial expansion of the left-hand side will be a multiple of the right-hand side).

Suppose

$$d = hcf(x, y).$$

Then

$$d^{x+y} | z^z \implies d | z,$$

since $z < x + y$. *Let*

$$x = dX, \ y = dY, \ z = dZ.$$

Then (taking the dth root on each side)

$$d^e X^X Y^Y = Z^Z,$$

where $e = X + Y - Z$.

In particular

$$X^X Y^Y | Z^Z.$$

Moreover, if we can find X, Y, Z *satisfying this relation and also satisfying*

$$X + Y - Z = 1$$

then dX, dY, dZ *will satisfy the original equation, with*

$$d = \frac{Z^Z}{X^X Y^Y}.$$

Suppose

$$u - v = 1.$$

Squaring,

$$u^2 + v^2 - 2uv = 1.$$

Let us therefore take

$$X = u^2, \ Y = v^2, \ Z = 2uv.$$

Then

$$\begin{aligned} \frac{Z^Z}{X^X Y^Y} &= 2^{2uv} u^{2uv-2u^2} v^{2uv-2v^2} \\ &= \frac{2^{2uv} v^{2v}}{u^{2u}}. \end{aligned}$$

We can make this to be an integer—as we want—by taking $u = 2^a$. In this case 2 occurs to the power $2uv$ on the top, and $2au$ on the bottom. Thus the condition will be satisfied if

$$v = 2^a - 1 \geq a.$$

This holds for all a. However, we only get a non-trivial solution if $a \geq 2$.

So we have the family of solutions

$$X = 2^{2a},\ Y = \left(2^a - 1\right)^2,\ Z = 2^{a+1}\left(2^a - 1\right)$$

for $a = 2, 3, \ldots$.

Taking $a = 2$ this gives

$$X = 2^4 = 16,\ Y = 3^2 = 9,\ Z = 2^3 3 = 24.$$

In this case $d = 2^8 3^6$, and so

$$x = 2^{12} 3^6,\ y = 2^8 3^8, z = 2^{11} 3^7.$$

9. If a, b, c are the lengths of the side of a triangle prove that

$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca),$$

and in each case find the necessary and sufficient condition.

Answer: *Taking the inequality on the left first, we have*

$$\begin{aligned} (a + b + c)^2 - 3(ab + bc + ca) &= a^2 + b^2 + c^2 - (ab + bc + ca) \\ &= \frac{1}{2}(a - b)^2 + \frac{1}{2}(b - c)^2 + \frac{1}{2}(c - a)^2. \end{aligned}$$

Thus the inequality holds, with equality if and only if $a = b = c$, ie the triangle is equiangular.

The inequality on the right can be written

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 0.$$

Now

$$ab + bc = b(a + c) \geq b^2,$$

since $a + c \geq b$. Similarly

$$bc + ca \geq c^2, \; ca + ab \geq a^2.$$

Adding the 3 inequalities,

$$2(ab + bc + ca) \geq a^2 + b^2 + c^2.$$

Now $a = b + c$ only if the 3 vertices A, B, C lie on a line with A between B and C. Thus the 3 inequalities cannot all be equalities simultaneously unless the 3 vertices coincide, ie $a = b = c = 0$.

10. Let S be the set of all natural numbers, which can be written in the form $x^3 + y^3 + z^3 - 3xyz$ for some $x, y, z \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. if $a \in S$ and $b \in S$ prove that $ab \in S$.

    Answer: *We have*

    $$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

    where $\omega = e^{2\pi i/3}$.

    Now if a, b, c are integers then

    $$a + b\omega + c\omega^2 = 0 \iff a = b = c \iff a + b\omega^2 + c\omega = 0.$$

    Suppose

    $$m = x^3 + y^3 + z^3 - 3xyz, \; n = X^3 + Y^3 + Z^3 - 3XYZ.$$

    We can certainly satisfy the equation

    $$u + \omega v + \omega^2 w = (x + \omega y + \omega^2 z)(X + \omega Y + \omega^2 Z)$$

    by taking

    $$u = xX + yZ + zY, \; v = xY + yX + zZ, \; w = xZ + yY + zX.$$

    Moreover, by the argument above this will also imply that

    $$u + \omega^2 v + \omega w = (x + \omega^2 y + \omega z)(X + \omega^2 Y + \omega Z)$$

    And it is easy to see that

    $$u + v + w = (x + y + z)(X + Y + Z).$$

    Multiplying these factors together,

    $$mn = u^3 + v^3 + w^3 - 3uvw.$$

    Thus $mn \in S$.