Irish Intervarsity Mathematics Competition 2008

National University of Ireland, Maynooth

11.00–14.00 Saturday 19$^{\text{th}}$ April 2008

1. Consider a regular n-gon inscribed inside a circle of radius 1. Each of the vertices of the n-gon lies on the circle. Draw a line between each pair of the n vertices. Prove that the sum of the squares of the lengths of these lines in n^2.

2. There are 63 coins, all identical in appearance, and all identical in weight except that one coin is slightly heavier than the others.

3. Given positive numbers $a_1 \geq a_2 \geq a_3$, $b_1 \geq b_2 \geq b_3$ and $c_1 \geq c_2 \geq c_3$, and three permutations α, β, γ of the set $\{1, 2, 3\}$, prove that

$$a_1b_1c_1 + a_2b_2c_2 + a_3b_3c_3 \geq a_{\alpha(1)}b_{\beta(1)}c_{\gamma(1)} + a_{\alpha(2)}b_{\beta(2)}c_{\gamma(2)} + a_{\alpha(3)}b_{\beta(3)}c_{\gamma(3)}.$$

4. Let $F(n)$ denote the usual Fibonacci sequence, which is defined by the properties $F(1) = F(2) = 1$ and $F(n+2) = F(n+1) + F(n)$ for $n > 0$. Let $(a(n))$ denote the *iterated Fibonacci sequence*, which is given by the equation $a(n) = F(F(n))$ for each $n > 0$. Prove that $a(n)$ is a multiple of 144 whenever $a(n)$ is a multiple of 14.

5. Take a positive integer N that ends in the digit 2. Move the 2 to the beginning of the number, and subtract 1 to get a new number, M. For example, if $N = 52$ then we move the 2 to the beginning of N to get 25, and then subtract 1 to obtain $M = 24$. What is the smallest value of N such that $M = 2N$?

6. For $n = 1, 2, \ldots$ let

$$a_n = n(n+1), \quad b_n = n+1.$$

What is the value of the convergent sequence

$$\frac{a_1}{b_1}, \frac{a_1}{b_1 + \frac{a_2}{b_2}}, \frac{a_1}{b_1 + \frac{a_2}{b_2 + \frac{a_3}{b_3}}}, \ldots ?$$

7. Let a triangle ABC have sides of length $|AB| = 5, |BC| = 6, |CA| = 7$, and let the lengths of the perpendiculars from a point P inside the triangle to the sides AB, BC and CA be x, y and z, respectively. Find the minimum value of the sum $x^2 + y^2 + z^2$.

8. Let (a_n) be a sequence of non-negative integers such that

$$a_{n+2} = \binom{a_{n+1}}{a_n},$$

where

$$\binom{m}{k} = \begin{cases} \frac{m!}{k!(m-k)!} & \text{if } m \geq k \\ 0 \text{ otherwise} \end{cases}$$

9. Show that there does not exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ which satisfies the following equation for all real values x:

$$f'(x) \geq 1 + |f(x)|^2.$$

10. In a room there are 2008 bulbs and 2008 buttons, and both are numbered from 1 to 2008. For each integer $1 \leq i \leq 2008$, pressing Button i changes the on or off status of Bulb i and one other bulb (the same bulb each time). Assuming that all bulbs are initially off, prove that by pressing the appropriate combination of buttons we can turn on at least 1340 of them simultaneously on. Prove also that if the effects of the buttons are chosen appropriately, 1340 is the largest number of bulbs that can be simultaneously on.