Irish Intervarsity Mathematics Competition 2006

University College Cork

9.30–12.30 Saturday 4th March 2006

1. Let $[x]$ denote the greatest integer not exceeding the real number x. Prove that
$$[\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}] = [\sqrt{9n+8}]$$
for all positive integers n.

2. The sides BC, CA, AB of the triangle ABC have lengths a, b, c, respectively, and satisfy
$$b^2 - a^2 = ac, \quad c^2 - b^2 = ab.$$
Determine the measure of the angles of ABC.

3. Prove that the polynomial
$$x^{100} + 33x^{67} + 67x^{33} + 101$$
cannot be factored as the product of two polynomials of lower degree with integer coefficients.

4. Let S be the set of all complex numbers of the form $a + bi$, where a and b are integers and $i = \sqrt{-1}$. Let A be a 2×2 matrix with entries in S and suppose A has determinant 1 and that
$$A^n = I.$$
for some positive integer n, where I is the identity matrix.
Prove that
$$A^{12} = I.$$

5. Evaluate
$$\int_0^{\pi/2} x\cot(x)\,dx.$$

6. Let x and y be positive integers and suppose that

$$k = \frac{x^2 + y^2 + 6}{xy}$$

is also an integer. Prove that k is the cube of an integer.