Irish Intervarsity Mathematics Competition 2006

University College Cork

9.30–12.30 Saturday 4th March 2006

1. Let $[x]$ denote the greatest integer not exceeding the real number x. Prove that
$$[\sqrt{n} + \sqrt{n+1} + \sqrt{n+2}] = [\sqrt{9n+8}]$$
for all positive integers n.

Answer: *Replacing n by $n+1$, we have to show that*
$$[\sqrt{n-1} + \sqrt{n} + \sqrt{n+1}] = [\sqrt{9n-1}]$$
for all integers $n \geq 2$.

If $f(x) = \sqrt{x}$ then
$$f'(x) = \frac{1}{2}\frac{1}{\sqrt{n}}, \quad f''(x) = -\frac{1}{4}\frac{1}{\sqrt{n}^3}.$$

By the Mean Value Theorem
$$\sqrt{n-1} - \sqrt{n} = \frac{1}{2}\frac{1}{\sqrt{n-\theta}}, \quad \sqrt{n} - \sqrt{n+1} = \frac{1}{2}\frac{1}{\sqrt{n+\phi}},$$
where $0 < \theta, \phi < 1$. Hence
$$\begin{aligned}\sqrt{n-1} + \sqrt{n} + \sqrt{n+1} &= 3\sqrt{n} + \frac{1}{2}\left(\frac{1}{\sqrt{n-\theta} - \sqrt{n+\phi}}\right) \\ &= 3\sqrt{n} - \frac{1}{4}\frac{\theta+\phi}{\sqrt{n+\psi}^3},\end{aligned}$$
where $|\psi| < 1$.

Similarly
$$\begin{aligned}\sqrt{9n-1} &= 3\sqrt{n-1/9} \\ &= 3\sqrt{n} - \frac{1}{6}\frac{1}{\sqrt{n-\chi}},\end{aligned}$$

where $0 < \chi < 1/9$.

Now

$$\frac{1}{4}\frac{\theta+\phi}{\sqrt{n+\psi}^3} < \frac{1}{2}\,\frac{1}{\sqrt{n-1}^3},$$

while

$$\frac{1}{6}\frac{1}{\sqrt{n-\chi}} > \frac{1}{6}\frac{1}{\sqrt{n}}.$$

But

$$\frac{1}{2}\,\frac{1}{\sqrt{n-1}^3} \le \frac{1}{6}\frac{1}{\sqrt{n}}$$

if

$$3 \ge n-1,$$

ie

$$n \ge 4.$$

Thus

$$\sqrt{9n-1} \le \sqrt{n-1} + \sqrt{n} + \sqrt{n+1}$$

if $n \ge 4$,

Note that

$$3\sqrt{n} < \sqrt{9n+8} < 3\sqrt{n+1}.$$

For positive x, *the function* $f(x) = \sqrt{x}$ *is convex upwards, since* $f''(x) < 0$. *Hence*

$$\sqrt{n} + \sqrt{n+2} < 2\sqrt{n+1}.$$

Thus

$$3\sqrt{n} < \sqrt{n} + \sqrt{n+1} + \sqrt{n+2} < 3\sqrt{n}.$$

So the two nearest integers must be equal unless

$$[3\sqrt{n+1}] > [3\sqrt{n}],$$

ie

$$[\sqrt{9n+9}] > [\sqrt{9n}].$$

This can only happen if $9n$ *is near to a square.*

More precisely, if

$$(m-1)^2 \le 9n < m^2$$

then

$$[\sqrt{9n+9}] = [\sqrt{9n}]$$

unless

$$9n = m^2 - r,$$

where $0 \le r \le 9$.

2. The sides BC, CA, AB of the triangle ABC have lengths a, b, c, respectively, and satisfy

$$b^2 - a^2 = ac, \quad c^2 - b^2 = ab.$$

Determine the measure of the angles of ABC.

Answer: *We have*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

etc.

Let

$$\frac{b}{a} = t.$$

Then

$$t^2 - 1 = c/a, \quad c^2/a^2 = t^2 + t.$$

Thus

$$(t^2 - 1)^2 = t^2 + t.$$

Taking out the factor $t + 1$,

$$(t - 1)(t^2 - 1) = t,$$

ie

$$t^3 - t^2 - 2t + 1 = 0.$$

Also

$$\frac{c}{a} = t^2 - 1, \quad \frac{c^2}{a^2} = t^2 + t.$$

Thus

$$\begin{aligned}\cos A &= \frac{t^2 + t^2 + t - 1}{2t^2(t+1)} \\ &= \frac{2t - 1}{2t^2}.\end{aligned}$$

Similarly

$$\begin{aligned}\cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{1 + t^2 + t - t^2}{2(t^2 - 1)} \\ &= \frac{t + 1}{2(t^2 - 1)},\end{aligned}$$

and

$$\begin{aligned}\cos C &= \frac{a^2+b^2-c^2}{2ab}\\ &= \frac{1+t^2-t^2-t}{2t}\\ &= \frac{1-t}{2t}.\end{aligned}$$

3. Prove that the polynomial

$$x^{100} + 33x^{67} + 67x^{33} + 101$$

cannot be factored as the product of two polynomials of lower degree with integer coefficients.

Answer: *Suppose the polynomial does factorise, say*

$$f(x) = u(x)v(x),$$

where $u(x), v(x)$ are monomial (have leading coefficients 1). Then the constant coefficients in $u(x), v(x)$ have product 101, and so are $\pm 1, \pm 101$.

We may suppose that $u(x)$ has constant coefficient ± 1. Then the product of the roots of $u(x)$ is ± 1, and so some root α has

$$|\alpha| \le 1.$$

But then

$$|\alpha^{100} + 33\alpha^{67} + 67\alpha^{33}| \le 1 + 33 + 67 = 101,$$

with equality only if $|\alpha| = 1$ and

$$\alpha^{100} = \alpha^{67} = \alpha^{33}.$$

In fact, if

$$\alpha^{100} + 33\alpha^{67} + 67\alpha^{33} + 101 = 0$$

then

$$\alpha^{100} = \alpha^{67} = \alpha^{33} = -1.$$

But

$$\alpha^{33} = -1 \implies \alpha^{99} = -1.$$

and

$$\begin{aligned}\alpha^{100} = -1, \alpha^{99} = -1 &\implies \alpha = 1\\ &\implies \alpha^{33} = 1,\end{aligned}$$

which is a contradiction.

4. Let S be the set of all complex numbers of the form $a + bi$, where a and b are integers and $i = \sqrt{-1}$. Let A be a 2×2 matrix with entries in S and suppose A has determinant 1 and that

$$A^n = I.$$

for some positive integer n, where I is the identity matrix.

Prove that

$$A^{12} = I.$$

Answer: *The eigenvalues ω_1, ω_2 of A must satisfy*

$$\omega^n = 1.$$

Also

$$\omega_1 \omega_2 = \det A = 1.$$

Thus

$$\omega_2 = \omega_1^{-1} = \overline{\omega_1}.$$

Hence

$$\operatorname{tr} A = \omega_1 + \omega_2 \in \mathbb{R}.$$

Since

$$\operatorname{tr} A = a + d \in S,$$

it follows that

$$\operatorname{tr} A \in \mathbb{Z}.$$

Moreover,

$$|\operatorname{tr} A| \leq |\omega_1| + |\omega_2| = 2.$$

Hence

$$\operatorname{tr} A = 0, \pm 1, \pm 2.$$

Note that A is semisimple (diagonalisable) in all cases, since it satisfies the separable equation $x^n - 1 = 0$.

If $\operatorname{tr} A = 0$ *then*

$$\omega_2 = -\omega_1 = \omega_1^{-1} \implies \omega_1, \omega_2 = \pm i.$$

In this case

$$A^4 = I.$$

If $\operatorname{tr} A = \pm 2$ *then*

$$\omega_1 = \omega_2 = \pm 1,$$

and

$$A = \pm I,$$

so that

$$A^2 = I.$$

Finally, if $\operatorname{tr} A = \pm 1$ *then its characteristic equation is*

$$x^2 \mp x + 1 = 0.$$

Thus

$$\omega_1, \omega_2 = -\omega, -\omega^2 \textit{ or } \omega, \omega^2,$$

where

$$\omega = \frac{1 + \sqrt{-3}}{2} = e^{2\pi/3},$$

in which case,

$$A^6 = I.$$

So

$$A^{12} = I$$

in all cases.

5. Evaluate

$$\int_0^{\pi/2} x \cot(x)\, dx.$$

Answer: *Let* $x = 2t$. *Then*

$$\begin{aligned}
I &= 4 \int_0^{\pi/4} t \cot(2t) dt \\
&= 4 \int_0^{\pi/4} t \, \frac{\tan^2 t - 1}{2 \tan t} \, dt \\
&= 2 \int_0^{\pi/4} t \; \tan t \; dt - 2 \int_0^{\pi/4} t \; \cot t \; dt \\
&= 2 \int_0^{\pi/4} t \; \tan t \; dt - 2(I - \int_{\pi/4}^{\pi/2} t \; \cot t \; dt) \\
&= -2I + 2 \int_0^{\pi/4} t \; \tan t \; dt + 2 \int_{\pi/4}^{\pi/2} t \; \cot t \; dt.
\end{aligned}$$

Changing variable $t \mapsto \pi/2 - t$ in the second integral,

$$\begin{aligned} 3I &= 2\int_0^{\pi/4} t\ \tan t\ dt + 2\int_0^{\pi/4} (\pi/2 - t)\ \cot t\ dt \\ &= \pi \int_0^{\pi/4} \tan t\ dt \\ &= \pi[-\log\cos t]_0^{\pi/4} \\ &= -\pi\log(1/\sqrt{2}) \\ &= \frac{\pi\log 2}{2}. \end{aligned}$$

Hence

$$I = \frac{\pi\ \log 2}{6}.$$

6. Let x and y be positive integers and suppose that

$$k = \frac{x^2 + y^2 + 6}{xy}$$

is also an integer. Prove that k is the cube of an integer.

Answer: *We have*

$$x^2 + y^2 + 6 - kxy = 0.$$

Regarding this as a quadratic in x, from one solution (x, y) we get another integer solution (x_1, y) where

$$xx_1 = y^2 + 6,$$

ie

$$x_1 = \frac{y^2 + 6}{x}.$$

Similarly regarding the equation as a quadratic in y, from one solution (x, y_1) we get another integer solution (x, y_1) with

$$y_1 = \frac{x^2 + 6}{y}.$$

Now let (x, y) be a solution (for a given k) with minimal x, and minimal y for that x. Then

$$x \le y,$$

and

$$\frac{x^2+6}{y} \geq y,$$

ie

$$y^2 \leq x^2 + 6.$$

Thus

$$x^2 \leq y^2 \leq x^2 + 6.$$

Note that $\gcd(x, y) = 1$. *For if* p *is prime,*

$$p \mid x, y \implies p^2 \mid xy \implies p^2 \mid k.$$

In particular $x \neq y$ *unless* $x = 1$, *in which case*

$$y^2 + 7 = ky \implies y \mid 7 \implies y = 1,\ k = 8 \text{ or } y = 7,\ k = 8.$$

If $y \neq x$ *then* $y \geq x + 1$ *and so*

$$(x+1)^2 \leq x^2 + 6 \implies 2x \leq 5 \implies x = 1 \text{ or } 2.$$

We have already dealt with the case $x = 1$. *It only remains to consider* $x = 2$. *In this case*

$$y^2 \leq x^2 + 6 \text{ and } y > x \implies y = 3.$$

But $(x, y) = (2, 3)$ *does not give integral* k.

We conclude that there is only a solution when $k = 8$, *which is a cube.*