Irish Intervarsity Mathematics Competition 2003

University College Dublin

Time allowed: Three hours

1. Let
$$f(x) = x^3 + Ax^2 + Bx + C,$$
where A, B, C are integers. Suppose the roots of $f(x) = 0$ (in the field of complex numbers) are α, β, γ. Prove that if
$$|\alpha| = |\beta| = |\gamma| = 1$$
then
$$f(x) \mid (x^{12} - 1)^3.$$

2. Let n be a positive integer. Prove that when written in decimal form (in base 10),
$$\left(\sqrt{17} + 4\right)^{2n+1}$$
has at least n zeroes following the decimal point.

3. Find all integers n for which
$$n^4 - 16n^3 + 86n^2 - 176n + 169$$
is the square of an integer.

4. Consider the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$
in which each positive integer k is repeated k times. Prove that its n^{th} term is
$$\left[\frac{1 + \sqrt{8n - 7}}{2}\right],$$
where $[x]$ denotes the greatest integer not exceeding x.

5. Let ABC be an acute angled triangle and a, b, c the lengths of the sides BC, CA, AB, respectively. Let P be a point inside ABC, and let x, y, z be the lengths PA, PB, PC, respectively. Prove that

$$(x + y + z)^2 \geq \frac{a^2 + b^2 + c^2}{2}.$$

6. Let $ABCD$ be a convex quadrilateral with the lengths $AB = AC$, $AD = CD$ and angles $B\hat{A}C = 20°$, $A\hat{D}C = 100°$. Prove that the lengths $AB = BC + CD$.

7. Let S be a set of 30 positive integers less than 100. Prove that there exists a nonempty subset T of S such that the product of the elements of T is the square of an integer.

8. Let

$$f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e,$$

where a, b, c, d, e are integers, and suppose that $f(x) = 0$ has no integer roots. Suppose also that $f(x) = 0$ has roots α, β (in the field of complex numbers) with $\alpha + \beta$ an integer. Show that $\alpha\beta$ is an integer.

9. Let x be a real number with $0 < x < 1$. Let $\{a_n\}$ be a sequence of positive real numbers. Prove that the inequality

$$1 + xa_n \geq x^2 a_{n-1}$$

holds for infinitely many positive integers n.

10. Find the least positive integer n for which

$$m^n - 1 \text{ is divisible by } 10^{2003}$$

for all integers m with greatest common divisor $\gcd(m, 10) = 1$.