

Irish Intervarsity Mathematics Competition 2002

University College Dublin

Time allowed: Three hours

Calculators and Tables may be used.

1. The length, breadth and height of a closed rectangular box are all an integer number of centimetres. If the volume of the box in cubic centimetres is the same as the total surface area of the box in square centimetres, what is the maximum volume the box can have?

2. Evaluate

$$\int \frac{dx}{6x^5 + x}$$

3. If for each positive integer n

$$(1 + x)^n = c_0 + c_1x + \cdots + c_nx^n$$

evaluate

$$\sum_{i=1}^n (2i + 1)c_i.$$

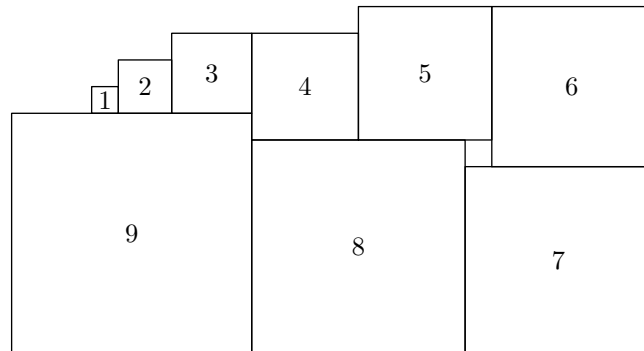
4. If a_i ; ($1 \leq i \leq n$), b are all real numbers, solve the equations

$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= b, \\ a_1^2 + a_2^2 + \cdots + a_n^2 &= b, \\ &\dots \\ a_1^n + a_2^n + \cdots + a_n^n &= b^n. \end{aligned}$$

5. If $f(n) = an^2 + bn + c$, where a, b, c and n are all positive integers, show that there exists a value of n for which $f(n)$ is not a prime number.
6. What is the area of a smallest rectangle into which squares of areas $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2$ and 9^2 can simultaneously be fitted without overlap?

7. Consider the curve given by the equation $y^2 = 4x$ in the plane. Two parallel chords are drawn, as in the diagram, giving line segments of lengths a, b, c and d . Prove that

$$a + d = b + c.$$



8. If x, y, z, w and t are positive integers such that

$$x^4 + y^4 + z^4 + w^4 = t^4,$$

prove that $xyzw$ is a multiple of 1000.

9. If a, b and c are positive integers such that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

what is the minimum possible value of ab ?

10. A sector of a plane circular disc of radius r and central angle θ is folded to give an open right circular cone. What value of θ , to the nearest degree, gives a cone of maximum volume?