Irish Intervarsity Mathematics Competition 2002

University College Dublin

Time allowed: Three hours

Calculators and Tables may be used.

1. The length, breadth and height of a closed rectangular box are all an integer number of centimetres. If the volume of the box in cubic centimetres is the same as the total surface area of the box in square centimetres, what is the maximum volume the box can have?

   Answer: *Let the length, breadth and height be a, b, c. Then*

$$V = abc,\ A = 2(ab + ac + bc).$$

Thus

$$abc = 2(ab + ac + bc).$$

We may suppose that

$$a \geq b \geq c.$$

Then

$$A \leq 6ab,$$

and so

$$abc \leq 6ab \implies c \leq 6.$$

Furthermore

$$c = 6 \implies ab = ac = bc \implies a = b = c = 6,$$

in which case

$$V = 6^3 = 216.$$

On the other hand,

$$A > 2ab \implies c > 2.$$

Thus we need only consider the cases $c = 3, 4, 5$.
If $c = 3$ *then*

$$ab = 6(a + b),$$

ie

$$a = \frac{6b}{b - 6}.$$

Thus

$$b \geq 7;$$

and since $a \geq b$,

$$\frac{6}{b - 6} \geq 1,$$

ie

$$b \leq 12.$$

The volume

$$V = abc = \frac{18b^2}{b - 6}.$$

The function

$$f(x) = \frac{x^2}{x - 6}$$

is stationary when

$$f'(x) = \frac{2x}{x - 6} - \frac{x^2}{(x - 6)^2} = 0,$$

ie

$$2(x - 6) = x,$$

ie

$$x = 12.$$

Thus $f(b)$ *is monotone on* $[7, 12]$. *Since*

$$\begin{aligned} b = 7 &\implies a = 42 \implies V = 42 \cdot 7 \cdot 3 = 882, \\ b = 12 &\implies a = 12 \implies V = 12 \cdot 12 \cdot 3 = 432, \end{aligned}$$

the maximum volume in this case is 882.

If $c = 4$ *then*

$$2ab = 8(a + b),$$

ie

$$a = \frac{4b}{b - 4}.$$

Thus

$$b \geq 5;$$

and since $a \geq b$,

$$\frac{4}{b - 4} \geq 1,$$

ie

$$b \leq 8.$$

The volume

$$V = abc = \frac{16b^2}{b - 4}.$$

The function

$$f(x) = \frac{x^2}{x - 4}$$

is stationary when

$$f'(x) = \frac{2x}{x - 4} - \frac{x^2}{(x - 4)^2} = 0,$$

ie

$$2(x - 4) = x,$$

ie

$$x = 8.$$

Thus $f(b)$ *is monotone on* $[5, 8]$. *Since*

$$b = 5 \implies a = 20 \implies V = 20 \cdot 5 \cdot 4 = 400,$$
$$b = 8 \implies a = 8 \implies V = 8 \cdot 8 \cdot 4 = 256,$$

the maximum volume in this case is 400.
Finally, if $c = 5$ *then*

$$3ab = 10(a + b),$$

ie

$$a = \frac{10b}{3b - 10}.$$

Thus

$$b \geq 4;$$

and since $a \geq b$*,*

$$\frac{10}{3b - 10} \geq 1,$$

ie

$$b \leq 6.$$

In fact a *is not integral if* $b = 6$*, while* $b \geq c = 5$*. Hence*

$$b = 5 \implies a = 10 \implies V = 10 \cdot 5 \cdot 5 = 250.$$

Thus the maximum volume is 882.

2. Evaluate

$$\int \frac{dx}{6x^5 + x}$$

Answer: *We have*

$$\begin{aligned} \frac{1}{6x^5 + x} &= \frac{1}{x(6x^4 + 1} \\ &= \frac{1}{x} - \frac{6x^3}{6x^4 + 1} \end{aligned}$$

Thus

$$\begin{aligned} I &= \int \frac{dx}{6x^5 + x} \\ &= \int \frac{1}{x}\, dx + \int \frac{6x^3}{6x^4 + 1}\, dx \\ &= \log x + \frac{1}{4} \log(6x^4 + 1). \end{aligned}$$

3. If for each positive integer n

$$(1+x)^n = c_0 + c_1 x + \cdots + c_n x^n$$

evaluate

$$\sum_{i=1}^{n} (2i+1) c_i.$$

Answer: *Substituting $x = 1$ in the given relation,*

$$\sum_{i=1}^{n} c_i = 2^n - 1.$$

Differentiating the given relation,

$$n(1+x)^{n-1} = \sum_{i=1}^{n} i c_i x^{i-1}$$

Substituting $x = 1$,

$$\sum_{i=1}^{n} i c_i = 2^{n-1} n.$$

Hence the given sum has value

$$2^n n + 2^n - 1 = 2^n(n+1) - 1.$$

4. If $a_i; (1 \leq i \leq n), b$ are all real numbers, solve the equations

$$\begin{gathered} a_1 + a_2 + \cdots + a_n = b, \\ a_1^2 + a_2^2 + \cdots + a_n^2 = b, \\ \cdots \\ a_1^n + a_2^n + \cdots + a_n^n = b^n. \end{gathered}$$

Answer: *From the theory of symmetric polynomials, we can express the symmetric products*

$$\begin{gathered} h_1 = \sum a_i, \\ h_2 = \sum_{i<j} a_i a_j, \\ \cdots h_n = a_1 a_2 \cdots a_n \end{gathered}$$

as polynomials in

$$s_1 = \sum a_i,$$
$$s_2 = \sum a_i^2,$$
$$\cdots s_n = \sum a_i^n.$$

But now $a_1, \ldots, a_n$ are the roots of the equation

$$t^n - h_1 t^{n-1} + h_2 t^{n-2} - \cdots + (-1)^n h_n = 0.$$

Thus if the roots of this equation are $\alpha_1, \ldots, \alpha_n$ then the solutions of the given equation are

$$\{a_1, \ldots, a_n\} = \{\alpha_1, \ldots, \alpha_n\}$$

in any order.

Evidently

$$(a_1, a_2, \ldots, a_n) = (b, 0, \ldots, 0)$$

is one solution to the equations. It follows from the argument above that the full set of solutions is obtained by permuting these values. Thus there are just n solutions:

$$(a_1, \ldots, a_n) = (0, \ldots, b, \ldots, 0).$$

5. If $f(n) = an^2 + bn + c$, where a, b, c and n are all positive integers, show that there exists a value of n for which $f(n)$ is not a prime number.

   Answer: *Suppose*

   $$f(a) = p.$$

   Then

   $$n \equiv a \bmod p \implies f(n) \equiv f(a) \equiv 0 \bmod p.$$

   Thus

   $$p \mid f(n),$$

   and so $f(n)$ is composite if $f(n) > p$ as it must be if n is large enough.

6. What is the area of a smallest rectangle into which squares of areas $1^2, 2^2, 3^2, 4^2, 5^2, 6^2, 7^2, 8^2 text and 9^2$ can simultaneously be fitted without overlap?

Answer: *The following answer is not very pretty. I don't know if there is a neater argument.*

Let us denote the squares by $1^2, \ldots, 9^2$. *The total area of the squares is*

$$1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81 = 285.$$

Let us say that two squares are x*-disjoint if their projections on the* x*-axis are disjoint; with* y*-disjointedness defined similarly. Then two squares must be either* x*-disjoint or* y*-disjoint.*

Let the width and height of the enclosing rectangle be w, h. *If the squares* $a_1^2, \ldots, a_r^2$ *are mutually* x*-disjoint then*

$$w \geq a_1 + \cdots + a_r.$$

Similarly if the squares are mutually y*-disjoint then*

$$h \geq a_1 + \cdots + a_r.$$

The squares 8^2 *and* 9^2 *are either* x*-disjoint or* y*-disjoint. We may suppose (swapping the coordinates if necessary) that they are* x*-disjoint.*

We consider two cases:

(a) 7^2 *is* x*-disjoint from both* 8^2 *and* 9^2;

(b) 7^2 *is* y*-disjoint from either* 8^2 *or* 9^2.

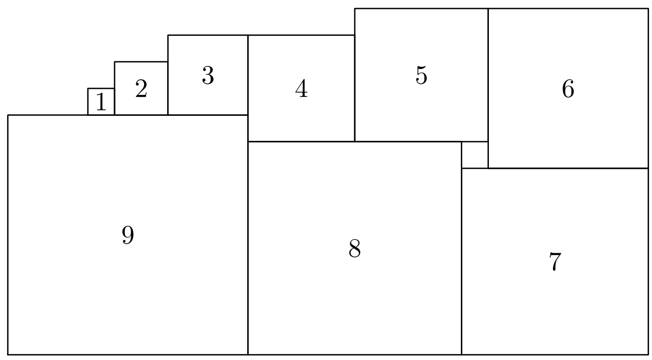
(a) *In the first case,*

$$w \geq 7 + 8 + 9 = 24.$$

The total width of the squares is

$$1 + 2 + \cdots + 9 = 45.$$

Thus the $7^2, 8^2, 9^2$ *squares take up more than half the width, and the remaining squares can be placed in a second row on top of them:*



With this arrangement

$$w = 24, \ h = 13;$$

and so the area of the enclosing rectangle is

$$A = 24 \cdot 13 = 312.$$

We can ignore arrangements leading to a larger rectangle.
We divide this case into two sub-cases:

i. 6^2 *is* y*-disjoint from one of* $7^2, 8^2, 9^2$*;*
ii. 6^2 *is* x*-disjoint from all of* $7^2, 8^2, 9^2$*.*

i. *In the first case*

$$h \geq 6 + 17 = 13;$$

and so

$$A \geq 24 \cdot 13 = 312.$$

So we can better the example above in this sub-case.

ii. *In the second case,*

$$w \geq 6 + 7 + 8 + 9 = 30.$$

Hence

$$h \leq 312/30,$$

ie

$$h \leq 10.$$

Thus 5^2 *cannot be* y*-disjoint to any of* $6^2, 7^2, 8^2, 9^2$*, since otherwise*

$$h \geq 5 + 6 = 11.$$

Hence $5^2, 6^2, 7^2, 8^2, 9^2$ *must be* x*-disjoint, and so*

$$w \geq 5 + 6 + 7 + 8 + 9 = 35,$$

and so

$$h \leq 312/35,$$

ie

$$h \leq 8,$$

which is impossible.

(b) Turning to the second case, in which 7^2 is y-disjoint to 8^2 or 9^2,

$$h \geq 7 + 8 = 15,$$

and so

$$w \leq 312/15,$$

ie

$$w \leq 20,$$

It follows that none of $4^2, 5^2, 6^2$ can be x-disjoint to both $8^2, 9^2$ since in that case

$$w \geq 4 + 8 + 9 = 21.$$

Hence each of these squares is y-disjoint to one of $8^2, 9^2$.

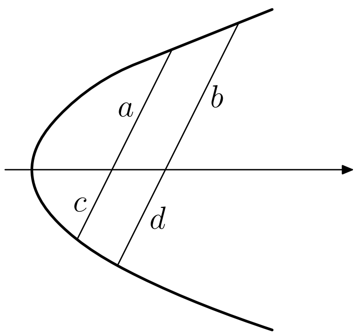
On the other hand, $4^2, 5^2, 6^2, 7^2$ cannot be mutually x-disjoint, since that would imply

$$w \geq 4 + 5 + 6 + 7 = 22.$$

It follows that two of these must be y-disjoint, as well as being y-disjoint to 8^2.

7. Consider the curve given by the equation $y^2 = 4x$ in the plane. Two parallel chords are drawn, as in the diagram, giving line segments of lengths a, b, c and d. Prove that

$$a + d = b + c.$$



Answer: *We represent points on the parabola parametrically:*

$$P(t) = (t^2, 2t).$$

Let the points A, B, C, D on the curve at the ends of the segments a, b, c, d have parameters $\alpha, \beta, \gamma, \delta$.

Let AC, BD meet the x-axis at

$$S = (\lambda, 0),\ T = (\mu, 0).$$

Then

$$\det \begin{pmatrix} \alpha^2 & 2\alpha & 1 \\ \lambda & 0 & 1 \\ \gamma^2 & 2\gamma & 1 \end{pmatrix} = 0,$$

ie

$$\lambda(\gamma - \alpha) = (\alpha^2\gamma - \gamma^2\alpha),$$

ie

$$\lambda = -\alpha\gamma.$$

Similarly,

$$\mu = -\beta\delta.$$

Thus

$$\begin{aligned} a^2 &= (\alpha^2 + \alpha\gamma)^2 + 4\alpha^2 \\ &= \alpha^2 \left((\alpha + \gamma)^2 + 4\right) \end{aligned}$$

8. If x, y, z, w and t are positive integers such that

$$x^4 + y^4 + z^4 + w^4 = t^4,$$

prove that $xyzw$ is a multiple of 1000.

Answer:

9. If a, b and c are positive integers such that

$$\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2},$$

what is the minimum possible value of ab?

Answer: *We have*

$$c^2(a^2 + b^2) = a^2b^2.$$

10. A sector of a place circular disc of radius r and central angle θ is folded to give an open right circular cone. What value of θ, to the nearest degree, gives a cone of maximum volume?

Answer: