

Irish Intervarsity Mathematics Competition 2001

Trinity College Dublin

9.30–12.30 Saturday 3rd March 2001

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.

Tables and calculators are not allowed.

1. For which integers a does the equation

$$x^3 - 3x + a$$

have three integer roots?

2. Find all solutions in integers a, b, c of

$$a^2 + b^2 + c^2 = a^2b^2.$$

3. Find all polynomials $P(x)$ satisfying

$$P(x^2) = P(x)P(x - 1).$$

4. If the plane is partitioned into two disjoint subsets, show that one of the subsets contains three points forming the vertices of an equilateral triangle.

More questions overleaf!

5. For which real numbers x is

$$\sin(\cos x) = \cos(\sin x)?$$

6. For which real numbers $c \neq 0$ does the sequence defined by

$$a_0 = c, \quad a_{n+1} = \frac{1}{2} \left(a_n + \frac{1}{a_n} \right)$$

converge?

7. Show that for any sequence a_n of strictly positive real numbers one (or both) of the series

$$\sum \frac{a_n}{n^2} \quad \text{and} \quad \sum \frac{1}{a_n}$$

must diverge.

8. The point A lies inside a circle centre O . At what point P on the circumference of the circle is the angle OPA maximised?
9. What is the maximum volume of a cylinder contained within a sphere of radius 1?
10. The function $f(x)$ is twice-differentiable for all x , and both $f(x)$ and $f''(x)$ are bounded. Show that $f'(x)$ is also bounded.