Irish Intervarsity Mathematics Competition 2001

Trinity College Dublin

9.30–12.30 Saturday 3rd March 2001

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. For which integers a does the equation

$$x^3 - 3x + a$$

have three integer roots?

Answer: *Suppose the roots are* u, v, w. *Then*

$$u + v + w = 0,\ uv + uw + vw = -3, uvw = -a.$$

Setting

$$w = -(u + v),$$

the second equation becomes

$$uv + uw + vw = -3,$$

ie

$$uv - (u + v)^2 = -3,$$

ie

$$u^2 + uv + v^2 = 3.$$

We have to find solutions $u, v \in \mathbb{Z}$ *of this equation. We can write the equation in the form*

$$(u + v/2)^2 + 3v^2/4 = 3.$$

Thus

$$3v^2/4 \leq 3,$$

ie

$$v^2 \leq 4,$$

ie

$$|v| \leq 2.$$

Similarly

$$|u| \leq 2.$$

On going through the possibilities

$$u, v \in \{-2, -1, 0, 1, 2\}$$

we see that the only integer solutions are

$$(u, v) = (-2, 1), (-1, -1), (1, 1), (2, -1).$$

Substituting for w*, the complete solutions are*

$$(u, v, w) = (-2, 1, 1), (-1, -1, 2), (1, 1, -2), (2, -1, -1).$$

Thus there are really just two solutions (up to order) namely

$$\pm(1, 1, -2)$$

giving

$$a = -uvw = \pm 2.$$

2. Find all solutions in integers a, b, c of

$$a^2 + b^2 + c^2 = a^2 b^2.$$

Answer: *One solution is evidently*

$$a = b = c = 0.$$

Suppose we have a different solution.

If $x \in \mathbb{Z}$ then

$$x^2 \equiv 0 \text{ or } 1 \bmod 4$$

according as x is even or odd.

Suppose a, b are both odd. Then

$$a^2 b^2 \equiv 1 \bmod 4,$$

while

$$a^2 + b^2 + c^2 \equiv 2 \text{ or } 3 \bmod 4.$$

Thus one (at least) of a, b is even. Hence

$$a^2 b^2 \equiv 0 \bmod 4,$$

and so

$$a^2 + b^2 + c^2 \equiv 0 \bmod 4,$$

which is only possible if a, b, c are all even.

Let 2^r be the highest power of 2 dividing a, b, c. Then $r \geq 1$ from above. Suppose

$$a = 2^r a', \ b = 2^r b', \ c = 2^r c'.$$

Then

$$a'^2 + b'^2 + c'^2 = 2^{2r} a'^2 b'^2.$$

Hence

$$a'^2 + b'^2 + c'^2 \equiv 0 \bmod 4,$$

which as we have seen implies that a', b', c' are all even, contrary to our choice of r as the highest power of 2 dividing a, b, c.

We conclude that

$$a = b = c = 0$$

is the only solution.

3. Find all polynomials $P(x)$ satisfying

$$P(x^2) = P(x)P(x-1).$$

Answer: *Suppose the roots of $P(x)$ are*

$$\alpha_1, \dots \alpha_n.$$

Then the roots of $P(x^2)$ are

$$\pm\sqrt{\alpha_1}, \dots \pm \sqrt{\alpha_n},$$

while the roots of $P(x)P(x-1)$ are

$$\alpha_1, \alpha_1 - 1, \dots, \alpha_n, \alpha_n - 1.$$

These must be the same, up to order.

Let

$$\max|\alpha_i| = M.$$

Then

$$\max|\sqrt{\alpha_i}| = \sqrt{M}.$$

Thus if $M > 1$ there is no way of matching the roots.

Similarly, let

$$\min_{\alpha_i \neq 0} |\alpha_i| = m.$$

Then

$$\min_{\alpha_i \neq 0} |\sqrt{\alpha_i}| = \sqrt{m}.$$

Thus if $m < 1$ there is no way of matching the roots.

We conclude that the non-zero roots must all lie on the unit circle:

$$\alpha_i \neq 0 \implies |\alpha_i| = 1.$$

Now suppose $\alpha \neq 0$ is a root of $P(x)$. Then $\alpha + 1$ is a root of $P(x-1)$. It follows that either $\alpha = -1$ or else

$$|\alpha| = 1, \ |\alpha + 1| = 1.$$

These two equations represent the two circles of radius 1 centred on 0 and -1, meeting in the points ω, ω^2 (where $\omega = \exp(2\pi i/3)$).

We conclude that if α is a roots of $P(x)$ then

$$\alpha \in \{0, -1, \omega, \omega^2\}.$$

If 0 *is a root of* $P(x)$ *then* -1 *is a root of* $P(x-1)$*, and so* $\pm i$ *are roots of* $P(x^2)$*. Hence* i *(for example) is a root of* $P(x)$ *or* $P(x-1)$*, ie* i *or* $i+1$ *is a root of* $P(x)$*, both of which we have already excluded.*

Similarly if -1 *is a root of* $P(x)$ *then* $\pm i$ *are roots of* $P(x^2)$*, which we have just seen is impossible.*

It follows that the only possible roots of $P(x)$ *are*

$$\alpha = \omega \text{ or } \omega^2.$$

Moreover these roots must occur equally often. For

$$\omega + 1 = -\omega^2,\ \omega^2 + 1 = -\omega;$$

and

$$\sqrt{\omega} = \pm\omega^2,\ \sqrt{\omega^2} = \pm\omega.$$

Thus if ω *occurs* r *times and* ω^2 *occurs* s *times as roots of* $P(x)$ *then* $\omega, -\omega, \omega^2, -\omega^2$ *occur with multiplicities* s, s, r, r *in* $P(x^2)$*, and with multiplicies* r, s, r, s *in* $P(x)P(x-1)$*.*

Since

$$(x-\omega)(x-\omega^2) = x^2 + x + 1,$$

we have shown that the only solutions of the given relation are

$$P(x) = (x^2 + x + 1)^n$$

for $n = 0, 1, 2, \ldots$ *(to which should be added the trivial solution* $P(x) = 0$*).*

4. If the plane is partitioned into two disjoint subsets, show that one of the subsets contains three points forming the vertices of an equilateral triangle.

   Answer: *There are probably many ways of doing this.*

   Let us call the two sets $\mathcal{A}$ *and* $\mathcal{B}$*. If* $\mathcal{A}$ *and* $\mathcal{B}$ *are empty the result is trivial; so we may assume that there are points* $A \in \mathcal{A},\ B \in \mathcal{B}$*.*

   Consider the line AB*, and equidistant points*

$$\ldots, P_{-2}, P_{-1}, A, B, P_1, P_2, \ldots$$

   on the line. There are two possibilities. Either these points are alternately in $\mathcal{A}$ *and* $\mathcal{B}$*,*

$$\ldots, A_{-1}, B_{-1}, A_0, B_0, A_1, B_1, \ldots;$$

or we can find three equidistant points X, Y, Z with the central point X in the same set as just one of the end-points X, Z, eg

$$A_1A_2B_1.$$

In the first case, consider the equilateral triangles

$$A_iX_iA_{i+1} \text{ and } A_iY_iA_{i+1},$$

with all the points X_i on the same side of the line. Then the points X_i, Y_i must all lie in $\mathcal{B}$; and so

$$X_1Y_2X_3$$

is an equilateral triangle in $\mathcal{B}$.

In the second case we may assume the three equidistant points are

$$A_1A_2B_1.$$

Consider the hexagon

$$A_1P_1P_2B_1P_3P_4$$

with centre A_2 and diagonal A_1B_1. Then $P_1 \in \mathcal{B}$, or $A_1P_1A_2$ would be an equilateral triangle in $\mathcal{A}$. Similarly $P_4 \in \mathcal{B}$. But then $P_1B_1P_4$ is an equilateral triangle in $\mathcal{B}$.

More questions overleaf!

5. For which real numbers x is

$$\sin(\cos x) = \cos(\sin x)?$$

Answer: *If*

$$\sin(\cos x) = \cos(\sin x),$$

then

$$\sin(\cos x) = \sin(\pi/2 - \sin x).$$

But

$$\sin A = \sin B \iff A = B + n\pi,$$

where $n \in \mathbb{Z}$. *Thus*

$$\cos x = n\pi + \pi/2 - \sin x.$$

Since

$$|\cos x|, |\sin x| \leq 1,$$

this is only possible if

$$n = 0 \text{ or } -1.$$

Thus

$$\cos x + \sin x = \pm\frac{\pi}{2}.$$

Squaring both sides,

$$\cos^2 x + 2\cos x \sin x + \sin^2 x = \frac{\pi^2}{4}.$$

Since

$$\cos^2 x + \sin^2 x = 1, \quad 2\cos x \sin x = \sin 2x,$$

our equation gives

$$\sin 2x = \frac{\pi^2}{4} - 1.$$

But $\pi^2 > 9$, *and so*

$$\frac{\pi^2}{4} - 1 > \frac{9}{4} - 1 > 1.$$

Since $|\sin 2x| \leq 1$ *for all* x, *we conclude that the original equation has no solution.*

6. For which real numbers $c \neq 0$ does the sequence defined by

$$a_0 = c, \quad a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$$

converge?

Answer: *Suppose the sequence converges to ℓ. Then*

$$\ell = \frac{1}{2}\left(\ell + \frac{1}{\ell}\right)$$

Hence

$$\ell = \frac{1}{\ell},$$

ie

$$\ell = \pm 1.$$

If c is replaced by $-c$ then a_n is replaced by $-a_n$ for all n. Thus it is sufficient to consider the case $c > 0$; if $a_n \to \ell$ when $a_0 = c > 0$ then $a_n \to -\ell$ when $a_0 = -c$.

If $c > 0$ then $a_n > 0$ for all n. Thus if a_n is convergent then $a_n \to 1$.

By the inequality between the arithmetic and geometric means (or by a simple calculation) if $a_n > 0$ then

$$\frac{1}{2}\left(a_n + \frac{1}{a_n}\right) \geq 1.$$

Thus

$$a_n \geq 1$$

for $n \geq 1$.

Also

$$\begin{aligned} a_{n+1} \leq a_n &\iff \frac{1}{a_n} \leq a_n \\ &\iff a_n \geq 1 \end{aligned}$$

Hence the sequence a_n is decreasing for $n \geq 1$, and so must converge to a limit, which we have seen must be 1.

We conclude that the sequence always converges if $c \neq 0$:

$$a_n \to \begin{cases} 1 & \text{if } c > 0, \\ -1 & \text{if } c < 0. \end{cases}$$

7. Show that for any sequence a_n of strictly positive real numbers one (or both) of the series
$$\sum \frac{a_n}{n^2} \quad \text{and} \quad \sum \frac{1}{a_n}$$
must diverge.

Answer: *Suppose the two series both converge. Then*
$$\frac{1}{a_n} \to 0 \implies a_n \to \infty.$$

Since $a_n > 0$ we can re-arrange the series without affecting their convergence or divergence. Thus we may assume that the a_n are increasing:
$$a_{n+1} \geq a_n.$$

Let $\pi(m)$ denote the number of a_n which are $\leq m$:
$$\pi(m) = |\{n : a_n \leq m\}|.$$

The number of a_n in the range $[m, 2m)$ is
$$\pi(2m) - \pi(m).$$

The a_n in this range contribute
$$\geq \frac{\pi(2m) - \pi(m)}{2m}$$
to the second series $\sum 1/a_n$.

It follows that
$$\frac{\pi(2m) - \pi(m)}{2m} \to 0$$
as $m \to \infty$. In particular there is an $N > 0$ such that
$$\frac{\pi(2m) - \pi(m)}{2m} < \frac{1}{8}$$
ie
$$\pi(2m) - \pi(m) < \frac{1}{4}m$$
for $m \geq N$.

Thus

$$\pi(2N) - \pi(N) \le \frac{1}{4}N,$$
$$\pi(4N) - \pi(2N) \le \frac{1}{4}2N,$$
$$\cdots$$
$$\pi(2^{r+1}N) - \pi(2^r 2N) \le \frac{1}{4}2^r N.$$

Adding,

$$\begin{aligned}\pi(2^{r+1}N) - \pi(N) &\le \frac{1}{4}\left(1 + 2 + 2^2 + \cdots + 2^r\right) N \\ &\le \frac{1}{4}2^{r+1}N\end{aligned}$$

Thus if

$$2^r N \le n < 2^{r+1}N$$

then

$$\begin{aligned}\pi(n) &\le \pi(2^{r+1}N) \\ &\le \pi(N) + \frac{1}{2}2^r N \\ &\le \pi(N) + \frac{1}{2}n.\end{aligned}$$

We conclude that, for some $N_1 \ge N$*,*

$$\pi(n) \le n$$

for $n \ge N_1$*.*

But

$$\pi(n) \le n \implies a_n \ge n.$$

Thus

$$\frac{a_n}{n^2} \ge \frac{1}{n}$$

for $n \ge N_1$*; and so the first series*

$$\sum \frac{a_n}{n^2}$$

diverges by comparison with

$$\sum \frac{1}{n}.$$

8. The point A lies inside a circle centre O. At what point P on the circumference of the circle is the angle OPA maximised?

   Answer: *It is evident that*

$$O\hat{P}A < \pi/2,$$

   eg by drawing the diameter D_1OAD_2 *through* A*, and comparing* $O\hat{P}A$ *with* $D_1PD_2 = \pi/2$.

   Applying the sine law to the triangle OPA,

$$\frac{\sin O\hat{P}A}{OA} = \frac{\sin O\hat{A}P}{OP}$$

   ie

$$\sin O\hat{P}A = \frac{OA}{OP} \sin O\hat{A}P.$$

   Now $O\hat{P}A$ *is maximised when* $\sin(O\hat{P}A)$ *is maximised. Since* OA *and* OP *are fixed (with* $OA < AP$*) this will occur when* $\sin(O\hat{A}P)$ *is maximised, ie when* $O\hat{A}P = \pi/2$.

   We conclude that $O\hat{P}A$ *is maximised when* P *is one of the two points where the line through* A *perpendicular to* OA *meets the circle.*

9. What is the maximum volume of a cylinder contained within a sphere of radius 1?

   Answer: *This is straightforward. Clearly the largest cylinder will have its centre at the centre of the sphere. Suppose the radius of the cylinder is* r*, and the height* h*. Then*

$$r^2 + h^2 = 1.$$

   The volume of the cylinder is

$$\begin{aligned} V &= \pi r^2 h \\ &= \pi h(1 - h^2). \end{aligned}$$

   We have to maximise this function, with the restriction that $0 < h < 1$.

   Let

$$f(x) = x(1 - x^2) = x - x^3.$$

   Then $f(0) = f(1) = 0$ *and*

$$f'(x) = 1 - 3x^2.$$

Thus $f(x)$ attains its maximum when

$$x = \frac{1}{\sqrt{3}};$$

and

$$V_{max} = \frac{\pi}{\sqrt{3}}.$$

10. The function $f(x)$ is twice-differentiable for all x, and both $f(x)$ and $f''(x)$ are bounded. Show that $f'(x)$ is also bounded.

Answer: *Suppose*

$$|f(x)| \le C, \quad |f''(x)| \le C.$$

Suppose $f'(x)$ is large at some point, say x_0. On taking $-f(x)$ in place of $f(x)$ if necessary, we may suppose that

$$f'(x_0) = X > 0.$$

Since $f''(x)$ is bounded, $f'(x)$ must remain large for a considerable interval beyond x_0. More precisely, suppose

$$f'(x) < X/2$$

for some $x > x_0$. Then

$$f'(x_0) - f'(x) > X/2.$$

But we know, from Rolle's Theorem, that

$$\frac{f'(x) - f'(x_0)}{x - x_0} = f''(\xi)$$

for some $\xi \in (x_0, x)$. Thus

$$\begin{aligned} x - x_0 &= \frac{f'(x) - f'(x_0)}{f''(\xi)} \\ &\ge \frac{X}{2C}. \end{aligned}$$

In other words,

$$f'(x) \ge X/2$$

for $x \in [x_0, x_1]$, where

$$x_1 = x_0 + \frac{X}{2C}.$$

But this in turn means that $f(x)$ is increasing steadily over this interval. More precisely, for some $\eta \in (x_0, x_1)$,

$$\begin{aligned} f(x_1) - f(x_0) &= (x_1 - x_0) f'(\eta) \\ &\geq (x_1 - x_0) \frac{X}{2}. \end{aligned}$$

Thus

$$f(x_1) - f(x_0) \geq \frac{X^2}{4C}.$$

But since

$$|f(x) \leq C$$

it follows that

$$|f(x_1 - f(x_0)| \leq |f(x_1|| + |f(x_0)| \leq 2C.$$

Hence

$$\frac{X^2}{4C} \leq 2C,$$

ie

$$X < 2^{3/2} C.$$

So we have shown that

$$|f'(x)| \leq 2^{3/2} C$$

for all x.