Maths Intervarsity Competition 2000

Dublin City University

10.00–13.00 March 4th

Answer all questions.

1. Prove that
$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$
for any positive integer n.

2. Determine *all* positive integers n for which $2^n + 1$ is divisible by 3.

3. Show that the sequence
$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7+\sqrt{7}}},\ \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}},\ldots$$
converges and evaluate the limit.

4. The sequence $\{x_0, x_1, x_2, \ldots\}$ is defined by the conditions
$$x_0 = a, \quad x_1 = b, \quad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n}.$$
Express $\lim_{n\to\infty} x_n$ concisely in terms of a and b (given numbers).

5. A random number generator can only select one of the nine integers $1, 2, \ldots, 9$ and it makes these selections with equal probability. Determine the probability that after n selections the product of the n numbers will be divisible by 10.

6. Evaluate
$$\int_{-\infty}^{\infty} e^{-ax^2 - bx^{-2}} dx, \qquad a, b > 0.$$

7. How many zeros are there at the end of the number $2000! = 1 \cdot 2 \cdot 3 \cdots 1999 \cdot 2000$?

8. It may seem odd but the sets $[0,1]$ and $[0,1)$ contain the same "numbers" of points. Find a one to one map of $[0,1]$ onto $[0,1)$.

9. The area T and an angle γ of a trianlge are given. Determine the lengths of the sides a and b so that the side cm opposite to the angle γ, is as short as possible.

10. What is the smallest amount that may be invested at interest rate i, compounded annually, in order that one may withdraw £1 at the end of the first year, £4 at the end of the second year, ..., £n^2 at the end of the n^{th} year, ..., in perpetuity?