Maths Intervarsity Competition 2000

Dublin City University

10.00–13.00 March 4th

Answer all questions.

1. Prove that

$$\sum_{k=1}^{n} \binom{n}{k}^2 = \binom{2n}{n},$$

for any positive integer n.

Answer: *Let*

$$c_k = \binom{n}{k}.$$

Then

$$c_k = c_{n-k};$$

and so the sum is

$$\sum_{k=1}^{n} c_k c_{n-k} = \textit{coefficient of } x^n \textit{ in } (1+c_1x+c_2x^2+\cdots+c_nx^n)(1+c_1x+c_2x^2+\cdots+c_nx^n)$$

But

$$(1 + c_1x + c_2x^2 + \cdots + c_nx^n) = (1 + x)^n.$$

Thus

$$\sum_{k=1}^{n} c_k c_{n-k} = \textit{coefficient of } x^n \textit{ in } (1 + x)^{2n} = \binom{2n}{n}.$$

2. Determine *all* positive integers n for which $2^n + 1$ is divisible by 3.

Answer: *Since*

$$2 \equiv -1 \bmod 3$$

it follows that

$$2^n \equiv (-1)^n \bmod 3$$

Hence

$$2^n + 1 \equiv (-1)^n + 1 \equiv \begin{cases} 2 \bmod 3 & \textit{if } n \textit{ is even} \\ 0 \bmod 3 & \textit{if } n \textit{ is odd} \end{cases}$$

Thus

$$3 \mid 2^n + 1$$

if and only if n is odd.

3. Show that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7+\sqrt{7}}},\ \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}}, \ldots$$

converges and evaluate the limit.

Answer: *Let*

$$a_0 = \sqrt{7},\ a_1 = \sqrt{7-\sqrt{7}}, \ldots.$$

Then

$$a_{n+2} = \sqrt{7-\sqrt{7+a_n}} \qquad (n \geq 0)$$

Suppose $a_n \to \ell$. Then

$$\ell = \sqrt{7-\sqrt{7+\ell}},$$

ie

$$\ell^2 = 7 - \sqrt{7+\ell},$$

ie

$$(7-\ell^2)^2 = \ell,$$

ie

$$\ell^4 - 14\ell^2 - \ell + 42 = 0.$$

One root of this is $\ell = 2$, *since* $16 - 56 - 2 + 42 = 0$. *Dividing,*

$$\ell^4 - 14\ell^2 - \ell + 42 = (\ell - 2)(\ell^3 + 2\ell^2 - 10\ell - 21).$$

Now we see that $\ell = -3$ *is also a root, since* $-27 + 18 + 30 - 21 = 0$. *Thus*

$$\ell^4 - 14\ell^2 - \ell + 42 = (\ell - 2)(\ell + 3)(\ell^2 - \ell - 7).$$

The full set of roots is

$$2,\ -3,\ \frac{1}{2}(1 \pm \sqrt{29}).$$

Now $a_n < \sqrt{7}$ *for all* n, *and so* $\ell \leq \sqrt{7}$. *Thus the only possible limit is* $\ell = 2$.

We observe that

$$\begin{aligned} a_n \leq \sqrt{7} &\implies \sqrt{7 + a_n} \leq \sqrt{7 + \sqrt{7}} \\ &\implies a_{n+2} \geq \sqrt{7 - \sqrt{7 + \sqrt{7}}} = a_2. \end{aligned}$$

Also

$$\begin{aligned} a_n \geq a_2 &\implies \sqrt{7 + a_n} \geq \sqrt{7 + a_2} \\ &\implies a_{n+2} \leq \sqrt{7 - \sqrt{7 + a_2}} = a_4. \end{aligned}$$

Thus

$$a_2 \leq a_n \leq a_4$$

for $n \geq 2$.

To prove that the sequence converges, note that

$$7 - a_{n+2}^2 = \sqrt{7 + a_n};$$

and so

$$(a_{n+2}^2 - 7)^2 = 7 + a_n.$$

Similarly,

$$(a_{n+3}^2 - 7)^2 = 7 + a_{n+1}.$$

Subtracting,

$$\begin{aligned} a_{n+1} - a_n &= a_{n+3}^4 - a_{n+2}^4 - 14(a_{n+3}^2 - a_{n+2}^2) \\ &= (a_{n+3}^2 - a_{n+2}^2)(a_{n+3}^2 + a_{n+2}^2 - 14) \\ &= (a_{n+3} - a_{n+2})(a_{n+3} + a_{n+2})(a_{n+3}^2 + a_{n+2}^2 - 14) \end{aligned}$$

Thus

$$\frac{a_{n+1} - a_n}{a_{n+3} - a_{n+2}} = -(a_{n+2} + a_{n+3})(14 - a_{n+2}^2 - a_{n+3}^2).$$

Now

$$a_{n+2} + a_{n+3} \geq 2a_2 \geq 2,$$

while

$$\begin{aligned} 14 - a_{n+2}^2 - a_{n+3}^2 &\geq 14 - 2a_4^2 \\ &= 14 - 2(7 - \sqrt{7 + \sqrt{a_2}}) \\ &= 2(\sqrt{7 + \sqrt{a_2}} \\ &> 4. \end{aligned}$$

Thus

$$|\frac{a_{n+1} - a_n}{a_{n+3} - a_{n+2}}| > 8 > 2^2.$$

It follows by induction that

$$|a_{n+3} - a_{n+2}| \leq \begin{cases} 2^{-n}|a_1 - a_0| & \textit{if } n \textit{ is odd} \\ 2^{-n}|a_2 - a_1| & \textit{if } n \textit{ is even.} \end{cases}$$

In all cases,

$$|a_{n+3} - a_{n+2}| \leq C2^{-n},$$

for some $C > 0$. Hence

$$a_n = a_0 + (a_1 - a_0) + \cdots + (a_n - a_{n-1})$$

is convergent.

4. The sequence $\{x_0, x_1, x_2, \ldots\}$ is defined by the conditions

$$x_0 = a, \quad x_1 = b, \quad x_{n+1} = \frac{x_{n-1} + (2n-1)x_n}{2n}.$$

Express $\lim_{n\to\infty} x_n$ concisely in terms of a and b (given numbers).

Answer: *We can write the given relation in the form*

$$x_{n+1} - x_n = -\frac{1}{2n}(x_n - x_{n-1}).$$

Thus by induction

$$\begin{aligned} x_{n+1} - x_n &= (-1)^n \frac{1}{2n \cdot 2(n-1) \cdots 2 \cdot 1}(x_1 - x_0) \\ &= (-1)^n \frac{1}{n!} 2^{-n}(x_1 - x_0) \end{aligned}$$

Hence

$$\begin{aligned} x_n &= x_0 + (x_1 - x_0) + (x_2 - x_1) + \cdots + (x_n - x_{n-1}) \\ &= x_0 + (x_1 - x_0)(1 - \frac{1}{2!}2^{-1} + \cdots \pm \frac{1}{n!}2^{-n}); \end{aligned}$$

and so

$$x_n \to x_0 + (x_1 - x_0)e^{1/2} = a + (b-a)e^{1/2}.$$

5. A random number generator can only select one of the nine integers $1, 2, \ldots, 9$ and it makes these selections with equal probability. Determine the probability that after n selections the product of the n numbers will be divisible by 10.

   Answer: *The product will be divisible by 10 if and only if 2 and 5 occur among the chosen numbers.*

   Let X denote the set of all 9^n choices, let S denote the set of choices not including 2, and T the set of choices not including 5. Then the set of choices containing at least one 2 and at least one 5 is

$$X \setminus (S \cup T).$$

Now

$$\|X\| = 9^n,$$

while

$$\|S\| = \|T\| = 8^n,$$

since in either case we have n choices from 8 digits; and similarly

$$\|S \cap T\| = 7^n,$$

since there are now 7 choices for each of the n digits.

Moreover,

$$\|S \cap T\| + \|S \cup T\| = \|S\| + \|T\|.$$

Thus

$$\|S \cup T\| = 2 \cdot 8^n - 7^n.$$

Hence

$$\|X \setminus (S \cup T)\| = 9^n - \|S \cup T\| = 9^n - 2 \cdot 8^n + 7^n;$$

and the probability that the choice lies in this set is

$$\frac{9^n - 2 \cdot 8^n + 7^n}{9^n} = 1 - 2(8/9)^n + (7/9)^n.$$

6. Evaluate

$$\int_{-\infty}^{\infty} e^{-ax^2 - bx^{-2}} dx, \qquad a, b > 0.$$

Answer: *Let*

$$I(a,b) = \int_{-\infty}^{\infty} e^{-(ax^2 + bx^{-2})} dx. \tag{$*$}$$

Substituting $x = \rho y$,

$$\begin{aligned} I(a,b) &= \int_{-\infty}^{\infty} e^{-(a\rho^2 x^2 + b\rho^{-2} x^{-2})} \rho dx \\ &= \rho I(\rho^2 a, \rho^{-2} b). \end{aligned}$$

Setting $\rho = \sqrt[4]{b/a}$,

$$I(a,b) = \sqrt[4]{b/a}\; I(\sqrt{ab}, \sqrt{ab}).$$

Let

$$J(c) = I(c,c).$$

Then

$$I(a,b) = \sqrt[4]{b/a}\; J(\sqrt{ab}).$$

Differentiating $(*)$ with respect to b 'under the integral sign',

$$\frac{\partial I(a,b)}{\partial b} = -\int_{-\infty}^{\infty} x^{-2} e^{-(ax^2 + bx^{-2})} dx.$$

(This is valid, since the integral converges uniformly absolutely.)
Substituting $x = y^{-1}$,

$$\begin{aligned} I(a,b) &= 2\int_0^\infty e^{-(ax^2+bx^{-2})}dx \\ &= 2\int_\infty^0 e^{-(ay^{-2}+by^2)} - \frac{dy}{y^2} \\ &= 2\int_{-\infty}^\infty x^{-2}e^{-(bx^2+ax^{-2})}dx. \end{aligned}$$

Thus

$$\frac{\partial I(a,b)}{\partial b} = -I(b,a).$$

But

$$I(a,b) = a^{-1/4}b^{1/4}J(\sqrt{ab}).$$

Hence

$$\frac{\partial I(a,b)}{\partial b} = \frac{1}{4}a^{-\frac{1}{4}}b^{-\frac{3}{4}}J(\sqrt{ab}) + \frac{1}{2}a^{-\frac{1}{4}}b^{-\frac{1}{4}}J'(\sqrt{ab}).$$

Thus

$$\frac{1}{4}a^{-\frac{1}{4}}b^{-\frac{3}{4}}J(\sqrt{ab}) + \frac{1}{2}a^{-\frac{1}{4}}b^{-\frac{1}{4}}J'(\sqrt{ab}) = -a^{\frac{1}{4}}b^{-\frac{1}{4}}J(\sqrt{ab}),$$

ie

$$J(\sqrt{ab})(4\sqrt{ab}+1) = -2\sqrt{ab}J'(\sqrt{ab}).$$

Thus

$$J'(x) = -\frac{4x+1}{2x}J(x),$$

ie

$$\frac{J'(x)}{J(x)} = -2x - \frac{1}{2x}.$$

Hence

$$\log J(x) = -2x - \frac{1}{2}\log x + c,$$

ie

$$J(x) = \frac{C}{\sqrt{x}}e^{-2x}.$$

Thus

$$\begin{aligned} I(a,b) &= a^{-\frac{1}{4}} b^{\frac{1}{4}} J(\sqrt{ab}) \\ &= C a^{-\frac{1}{2}} e^{-2\sqrt{ab}}. \end{aligned}$$

In particular,

$$I(1,0) = C.$$

But we know that

$$I(1,0) = \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Hence $C = \sqrt{\pi}$*, and so*

$$I(a,b) = \sqrt{\tfrac{\pi}{a}} e^{-2\sqrt{ab}}.$$

7. How many zeros are there at the end of the number $2000! = 1 \cdot 2 \cdot 3 \cdots 1999 \cdot 2000$?

   Answer: *Suppose 2 occurs to power* m *in* $2000!$*, and 5 to power* n*. Then the number of zeros is* $\min(m,n)$*.*

   Now 2 *divides* $[2000/2]$ *of the numbers (where* $[x]$ *denotes the greatest integer* $\le x$*),* 2^2 *divides* $[2000/2^2]$ *of the numbers, etc. It follows that*

   $$m = [2000/2] + [2000/2^2] + [2000/2^3] + \cdots ;$$

   and similarly

   $$n = [2000/5] + [2000/5^2] + [2000/5^3] + \cdots ,$$

   It is clear that $n \le m$*, since every term in the second sum is* $\le$ *the corresponding term in the first sum.*

   Thus the number of zeros at the end of $2000!$ *is*

   $$n = 400 + 80 + 16 + 3 = 499.$$

8. It may seem odd but the sets $[0,1]$ and $[0,1)$ contain the same "numbers" of points. Find a one to one map of $[0,1]$ onto $[0,1)$.

   Answer: *The map*

   $$f : [0,1] \to [0,1)$$

   defined by

   $$f(x) = \begin{cases} \frac{1}{n+1} & \text{if } x = \frac{1}{n} \\ x & \text{if } x \text{ is not of the form } \frac{1}{n} \end{cases}$$

   has the required property. Thus f maps 1 to 1/2, 1/2 to 1/3, etc, and maps numbers not of this form to themselves.

9. The area T and an angle γ of a trianlge are given. Determine the lengths of the sides a and b so that the side cm opposite to the angle γ, is as short as possible.

   Answer: *We have*

   $$T = \frac{1}{2} ab \sin\gamma,$$

   while

   $$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ &= (a-b)^2 + 2ab(1-\cos\gamma) \\ &= (a-b)^2 + \frac{4T(1-\cos\gamma)}{\sin\gamma} \end{aligned}$$

   Thus c is minimized when $a = b$, in which case

   $$a = b = \sqrt{\frac{2T}{\sin\gamma}}.$$

10. What is the smallest amount that may be invested at interest rate i, compounded annually, in order that one may withdraw £1 at the end of the first year, £4 at the end of the second year, ..., £n^2 at the end of the n^{th} year, ..., in perpetuity?

    Answer: *Let the sum at the end of n years be $a(n)$ pounds. Then*

    $$a(n+1) = (1+\lambda)a(n) - n^2,$$

    where $\lambda = i/100$. Thus

    $$a(n+1) - (1+\lambda)a(n) = -n^2.$$

The solution of the homogeneous problem

$$a(n+1) - (1+\lambda)a(n) = 0$$

is

$$a(n) = \rho(1+\lambda)^n,$$

where ρ is a constant.

For a particular solution let us try

$$a(n) = An^2 + Bn + C.$$

Since

$$A(n+1)^2 + B(n+1) + C = (1+\lambda)(An^2 + Bn + C) - n^2$$

for all n,

$$\begin{aligned} \lambda A &= 1, \\ \lambda B &= 2A, \\ \lambda C &= A + B. \end{aligned}$$

Hence

$$A = \frac{1}{\lambda},\ B = \frac{2}{\lambda^2},\ C = \frac{\lambda+2}{\lambda^3}.$$

Thus the general solution is

$$a(n) = \rho(1+\lambda)^n + \frac{\lambda^2 n^2 + 2\lambda n + \lambda + 2}{\lambda^3}.$$

We see that $a(n) \geq 0$ for all n if and only if $\rho \geq 0$. But

$$a(0) = \rho + \frac{\lambda+2}{\lambda^3}.$$

Thus the smallest value that $a(0)$ can have is

$$\frac{\lambda+2}{\lambda^3} = \frac{200+i}{i^3} \times 10000$$

pounds.