

Irish Intervarsity Mathematics Competition 1999

University College Cork

Answer all questions. Tables and calculators may be used.

1. Find the value of

$$\lim_{n \rightarrow \infty} \left(\sum_{r=0}^n \frac{1}{\binom{n}{r}} \right),$$

where $\binom{n}{r}$ is the binomial coefficient, the number of combinations of n things taken r at a time.

2. The coordinates of four points in the plane are given by $A(0, 0)$, $B(1, 2)$, $C(3, 3)$ and $D(3, 0)$. What is the smallest possible value of $|PA| + |PB| + |PC| + |PD|$, where P is any point in the plane.
3. Two real numbers a and b are chosen with $0 \leq a \leq 1$ and $0 \leq b \leq 1$. What is the probability that $a^2 + b^2 \leq 1$.
4. If x, y, z, w, t and u are all prime numbers with $x \leq y \leq z \leq w \leq t \leq u$, find all solutions of the equation

$$x^2 + y^2 + z^2 + w^2 + t^2 = u^2.$$

5. Evaluate $\sum_{r=1}^n (r+1)^2 (r!)$.
6. Find all positive integers less than 100 which have precisely seven distinct divisors (including 1 and n).
7. Let ABC be a triangle with a right angle at A . Show that the internal bisector of the angle BAC divides the square on the hypotenuse $BCDE$ into two parts of equal area.

8. Evaluate

$$f(m, n) = \int_0^1 x^m(1-x)^n dx$$

as a function of m and n only.

9. A window of total perimeter 200cm consists of a rectangle surmounted by a semicircle. Find the maximal area the window can have.
10. The lengths of the sides of a quadrilateral are 1, 2, 3 and 4. What is the maximal area the quadrilateral can have?