# Irish Intervarsity Mathematics Competition 1999

University College Cork

*Answer* all *questions. Tables and calculators may be used.*

1. Find the value of
$$\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{\binom{n}{r}}\right),$$
where $\binom{n}{r}$ is the binomial coefficient, the number of combinations of $n$ things taken $r$ at a time.

   **Answer:** *Let us write $c_r$ for $\binom{n}{r}$. Then $c_{n-r} = c_r$; and the binomial coefficient $c_r$ increases as $r$ increases from $r = 0$ to the middle of $[0, n]$, and then decreases to $r = n$. In particular,*
$$c_r \geq c_2 = \frac{n(n-1)}{2} \text{ if } r \neq 0, 1, n-1, n.$$
   *Thus*
$$S(n) = \sum \frac{1}{c_r} \leq 1 + \frac{1}{n} + \delta + \frac{1}{n} + 1,$$
   *where*
$$\delta \leq n\frac{2}{n(n-1)} = \frac{2}{n-1}.$$
   *Thus each of the middle three parts tends to 0, leaving the two 1's at the ends, and so*
$$S(n) \to 2 \text{ as } n \to \infty.$$

   Remark: *I thought at first that one would have to use the estimate for $\binom{n}{r}$ which one gets from the Law of Large Numbers, ie for large $n$ the binomial coefficients approximate to the curve $e^{-x^2}$, suitably scaled and translated.*

2. The coordinates of four points in the plane are given by $A(0,0), B(1,2), C(3,3)$ and $D(3,0)$. What is the smallest possible value of $|PA|+|PA|+|PC|+|PD|$, where $P$ is any point in the plane.

**Answer:** Remark: *There were three "resources" that occurred to me for dealing with this question:*

(a) *The "reflection principle" that if $A, B$ are two points on the same side of the line $\ell$ then $AP + BP$ is minimised, for $P$ on $\ell$, when $AP$ and $BP$ make the same angle with $\ell$. (This is easily proved by joining $A$ to the reflection of $B$ in $\ell$.)*

(b) *The locus of points such that $PA + PB = c$, where $c$ is a constant, is an ellipse with foci at $A, B$.*

(c) *If $\boldsymbol{r}$ is a vector of length $r = |\boldsymbol{r}|$ then*

$$\begin{aligned} dr &= \frac{\boldsymbol{r}.d\boldsymbol{r}}{r} \\ &= \boldsymbol{u}.d\boldsymbol{r}, \end{aligned}$$

*where $\boldsymbol{u}$ is a unit vector in the direction of $\boldsymbol{r}$.*

*Now for the answer. We have $AB = BC = \sqrt{5}$, $CD = DA = 3$. Thus the quadrilateral splits into two isosceles triangles, and is symmetric about the diagonal $BD$. Also the diagonals are each at angle $\pi/4$ to the axes, and so are perpendicular.*

*If we assume the minimum occurs at a point $P$ on the axis of symmetry, then the result follows at once; for $BP + PD$ is constant for points on the segment $BD$, and $AP + PC$ is clearly minimised when $APC$ is a straight line. Hence the minimum occurs where the diagonals meet, ie at the mid-point of $AC$, namely $P = (3/2, 3/2)$.*

*To see that the minimum must occur on this line $BD$. suppose there was a minimum at a point $M$ off the line. Then the reflection $M'$ of $M$ in $BD$ is also a minimum. But consider $P$ on the segment $MM'$. By the "reflection principle" above, $AP + CP$ is minimised when $P$ is at the mid-point of $MM'$ (on the axis $BD$); and both $BP$ and $DP$ get smaller when $P$ moves from $M$ to this mid-point. Hence the minimum must occur on the axis of symmetry.*

3. Two real numbers $a$ and $b$ are chosen with $0 \le a \le 1$ and $0 \le b \le 1$. What is the probability that $a^2 + b^2 \le 1$.

**Answer:** *The point $(a, b)$ is evenly distributed over the unit square. Hence the probability is just the area of the subset $\{(a, b) : a^2 + b^2 \le 1\}$*

*of this square, ie the quarter-circle with radius* $1$*. Thus the probability is* $\pi/4$*.*

4. If $x, y, z, w, t$ and $u$ are all prime numbers with $x \le y \le z \le w \le t \le u$, find all solutions of the equation

$$x^2 + y^2 + z^2 + w^2 + t^2 = u^2.$$

**Answer:** *This is based on the fact that if* $x$ *is odd then*

$$x^2 \equiv 1 \bmod 8.$$

*Evidently* $u$ *is odd, since 2 is the only even prime. Thus*

$$u^2 \equiv 1 \bmod 8.$$

*Suppose* $a$ *of the numbers on the left are 2, and the remaining* $5-a$ *are odd. Then*

$$x^2 + y^2 + z^2 + w^2 + t^2 \equiv 4a + (5-a) \bmod 8.$$

*The only way in which this can be* $\equiv 1 \bmod 8$ *is if* $a = 4$ *and* $5 - a = 1$*. In other words the equation is*

$$2^2 + 2^2 + 2^2 + 2^2 + t^2 = u^2, \tag{1}$$

*ie*

$$16 + t^2 = u^2, \tag{2}$$

*ie*

$$16 = (u-t)(u+t). \tag{3}$$

*Since* $t$ *and* $u$ *are odd,* $u - t$ *and* $u + t$ *are both even. On factoring 16, there is only one possibility:*

$$(u-t, u+t) = (2, 8), \tag{4}$$

*ie*

$$t = 3,\ u = 5. \tag{5}$$

*so the only solution is*

$$2^2 + 2^2 + 2^2 + 2^2 + 3^2 = 5^2.$$

5. Evaluate $\sum_{r=1}^{n}(r+1)^2(r!)$.

**Answer:** Remark: *There are only two simple ways in which one might get a "closed" formula for a sum like this:*

*(a) by expressing the sum as a coefficient in the product of two polynomials; and*

*(b) by expressing the rth term in the form*

$$a_r = f(r) - f(r-1)$$

*for some function $f(r)$.*

*In this case we have*

$$\begin{aligned}(r+1)^2 r! &= (r+1)(r+1)! \\ &= (r+2)(r+1)! - (r+1)! \\ &= (r+2)! - (r+1)!\end{aligned}$$

*Thus*

$$\begin{aligned}\sum_{r=1}^{n}(r+1)^2 r! &= (3!-2!) + (4!-3!) + \cdots + ((n+2)! - (n+1)!) \\ &= (n+2)! - 2.\end{aligned}$$

6. Find all positive integers less than 100 which have precisely seven distinct divisors (including 1 and $n$).

**Answer:** *Suppose*

$$n = p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r},$$

*where $p_1, \ldots, p_r$ are distinct primes, and each $e_i$ is a positive integer. Then the factors of $n$ are the numbers*

$$d = p_1^{j_1} p_2^{j_2} \cdots p_r^{j_r},$$

*where*

$$0 \le j_i \le e_i$$

*for $i = 1, \ldots, r$. Thus $n$ has just*

$$(e_1+1)(e_2+1)\cdots(e_r+1)$$

*factors.*

*If this is 7, then we must have $r = 1$ and $e_1 = 6$, ie $n = p_1^6$. The only such power less than 100 is $2^6 = 64$. Thus there is only one number in this range with just 7 divisors.*

7. Let $ABC$ be a triangle with a right angle at $A$. Show that the internal bisector of the angle $BAC$ divides the square on the hypotenuse $BCDE$ into two parts of equal area.

**Answer:** *A line divides a rectangle into 2 equal parts if and only if it goes through the centre $O$. For if the line goes through the centre, then reflection in the centre (which is an isometry, and so preserves area) sends one part into the other; while if the line does not go through the centre then the parallel line through the centre divides the square into equal parts, so the original line cannot do this.*

*It follows that we have to show that the bisector of $A$ goes through the centre $O$ of the square $BCDE$.*

*The circle with BC as diameter passes through $A$ and $O$, since the angles $BAC$ and $BOC$ are both $\pi/2$. Hence*

$$CAO = CBO = \pi/4.$$

*Thus $CO$ bisects the right angle $A$, and the result follows.*

8. Evaluate

$$f(m, n) = \int_0^1 x^m (1-x)^n \, dx$$

as a function of $m$ and $n$ only.

**Answer:** *Integrating by parts,*

$$f(m, n) = \left[ \frac{x^{m+1}}{m+1} (1-x)^n \right]_0^1 + \int_0^1 \frac{x^{m+1}}{m+1} n(1-x)^{n-1}$$

*The first term vanishes, since $x^{m+1} = 0$ at $x = 0$ and $(1-x)^n = 0$ at $x = 1$ (assuming $n \geq 1$). Thus*

$$f(m, n) = \frac{n}{m+1} f(m+1, n-1).$$

*Repeating this,*

$$\begin{aligned}
f(m,n) &= \frac{n}{m+1} f(m+1,n-1) \\
&= \frac{n}{m+1}\frac{n-1}{m+2} f(m+2,n-2) \\
\cdots \\
&= \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n)} f(m+n,0) \\
&= \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n)} \int_0^1 x^{m+n}\,dx \\
&= \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n)} \frac{1}{m+n+1} \\
&= \frac{n(n-1)\cdots 1}{(m+1)(m+2)\cdots(m+n+1)}.
\end{aligned}$$

*Thus*

$$f(m,n) = \frac{m!n!}{(m+n+1)!}.$$

9. A window of total perimeter 200cm consists of a rectangle surmounted by a semicircle. Find the maximal area the window can have.

   **Answer:** *Let the semicircle have radius* $r$*, and the rectangle have sides* $2r, 2s$*. Then the perimeter is*

$$P = \pi r + 2r + 4s = (\pi+2)r + 4s,$$

*while the area is*

$$A = \pi r^2 + 4rs.$$

*It follows that*

$$dP = (\pi+2)\,dr + 4\,ds,$$

*while*

$$dA = (2\pi r + 4s)\,dr + 4r\,ds.$$

*At a stationary point,* $dA = 0$ *whenever* $dP = 0$*. It follows that the two linear forms in* $dr, ds$ *must be proportional, ie*

$$\frac{2\pi r + 4s}{\pi + 2} = \frac{4r}{4}.$$

*Thus*

$$(2\pi r + 4s) = (\pi + 2)r,$$

*and so*

$$s = \frac{\pi - 2}{4} r.$$

*Since* $P = (\pi + 2)r + 4s = 200$, *we conclude that*

$$r = \frac{100}{\pi}, \; s = \frac{25(\pi - 2)}{\pi},$$

*giving minimal area*

$$\begin{aligned} A &= \frac{10000}{\pi} + \frac{10000(\pi - 2)}{\pi^2} \\ &= 20000 \frac{\pi - 1}{\pi^2}. \end{aligned}$$

10. The lengths of the sides of a quadrilateral are $1, 2, 3$ and $4$. What is the maximal area the quadrilateral can have?

    **Answer:** *Suppose a quadrilateral has sides* $AB = a, BC = b, CD = c, DA = d$. *Then its area* $\Delta$ *is given by*

    $$2\Delta = ab \sin A + cd \sin C.$$

    *Let the diagonal* $AC = x$. *Then*

    $$x^2 = a^2 + b^2 - 2ab\cos A, x^2 = c^2 + d^2 - 2cd\cos C.$$

    *It follows that*

    $$2x\,dx = -2ab\sin A\,dA = -2cd\sin C\,dC.$$

    *On the other hand,*

    $$2\,dA = ab\cos A\,dA + cd\cos C\,dC$$

    *If the area is stationary, then* $d\Delta = 0$ *whenever* $ab\sin A\,dA = cd\sin C\,dC$. *In other words, the two linear forms in* $dA, dC$ *are proportional, ie*

    $$\frac{ab\sin A}{ab\cos A} = \frac{-cd\sin C}{cd\cos C},$$

*ie*

$$\tan A = -\tan C,$$

*ie*

$$A + C = \pi.$$

*Thus we have proved the theorem that* a quadrilateral with given sides has maximal area when it is concyclic.

*In the present case, our two equations for* $x^2$ *give*

$$a^2 + b^2 - 2ab\cos A = c^2 + d^2 - 2cd\cos C,$$

*which with* $\cos C = -\cos A$ *gives*

$$(2 \cdot 1 \cdot 2 + 2 \cdot 3 \cdot 4)\cos A = 1^2 + 2^2 - 3^2 - 4^2,$$

*ie*

$$\cos A = -\frac{20}{28} = -\frac{5}{7},$$

*and so*

$$\sin A = \sin C = \frac{\sqrt{24}}{7} = \frac{2}{7}\sqrt{6}.$$

*We conclude that the maximal area is*

$$\Delta = \frac{ab + cd}{2}\sin A = \frac{14}{2} \cdot \frac{2}{7}\sqrt{6} = 2\sqrt{6}.$$