Irish Intervarsity Mathematics Competition

Trinity College Dublin 1997

9.30–12.30 Saturday 22^{nd} February 1997

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. Compute
$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

2. A stick is broken in random in 2 places (the 2 break-points being chosen independently). What is the probability that the 3 pieces form a triangle?

3. For which real numbers $x > 0$ is there a real number $y > x$ such that
$$x^y = y^x \ ?$$

4. Show that there are an infinity of natural numbers n such that when the last digit of n is moved to the beginning (as eg $1234 \mapsto 4123$) n is multiplied by 3.

5. What is the whole number part of
$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{1997}} \ ?$$

6. Prove that
$$n(n+1)(n+2) > \left(n + \frac{8}{9}\right)^3$$
for any integer $n \geq 3$.

7. Show that the determinant of the 3×3 matrix

$$A = \begin{pmatrix} \sin(x_1 + y_1) & \sin(x_1 + y_2) & \sin(x_1 + y_3) \\ \sin(x_2 + y_1) & \sin(x_2 + y_2) & \sin(x_2 + y_3) \\ \sin(x_3 + y_1) & \sin(x_3 + y_2) & \sin(x_3 + y_3) \end{pmatrix}$$

is zero for all real numbers $x_1, x_2, x_3, y_1, y_2, y_3$.

8. Let

$$A(m, n) = \frac{m!(2m + 2n)!}{(2m)!n!(m + n)!}$$

for non-negative integers m and n. Show that

$$A(m, n) = 4A(m, n - 1) + A(m - 1, n)$$

for $m \geq 1,\ n \geq 1$. Hence or otherwise show that $A(m, n)$ is always an integer.

9. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real polynomial of degree $n \geq 2$ such that

$$0 < a_0 < -\sum_{k=1}^{[n/2]} \frac{1}{2k + 1} a_{2k}$$

(where $[n/2]$ denotes the integer part of $n/2$). Prove that the equation $P(x) = 0$ has at least one solution in the range $-1 < x < 1$.

10. Suppose $a_1, a_2, a_3, \ldots$ is an infinite sequence of real numbers satisfying $0 < a_n \leq 1$ for all n. Let $S_n = a_1 + a_2 + \cdots + a_n$ and $T_n = S_1 + S_2 + \cdots + S_n$. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{T_n} < \infty.$$