Irish Intervarsity Mathematics Competition

Trinity College Dublin 1997

9.30–12.30 Saturday 22nd February 1997

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. Compute
$$\sum_{n=1}^{\infty} \frac{n^2}{2^n}.$$

Answer: *First solution: Let*

$$\begin{aligned} f(x) &= \sum_0^{\infty} \frac{x^n}{2^n} \\ &= 1 + \frac{x}{2} + \frac{x^2}{2^2} + \cdots \\ &= \frac{1}{1 - x/2} \\ &= \frac{2}{2-x}. \end{aligned}$$

Differentiating,
$$\sum_1^{\infty} \frac{nx^{n-1}}{2^n} = \frac{2}{(2-x)^2}.$$

Multiplying by x,
$$\sum_1^{\infty} \frac{nx^n}{2^n} = \frac{2x}{(2-x)^2}.$$

Differentiating again,

$$\sum_{1}^{\infty} \frac{n^2 x^{n-1}}{2^n} = \frac{2}{(2-x)^2} + \frac{4x}{(2-x)^3}.$$

Substituting $x = 1$*,*

$$\sum_{1}^{\infty} \frac{n^2}{2^n} = 2 + 4 = 6.$$

Second solution: Let

$$S = \sum_{1}^{\infty} \frac{n^2}{2^n}$$

Then

$$\begin{aligned} 2S &= \sum_{1}^{\infty} \frac{n^2}{2^{n-1}} \\ &= \sum_{0}^{\infty} \frac{(n+1)^2}{2^n} \\ &= S + 2T + \sum_{0}^{\infty} \frac{1}{2^n}, \end{aligned}$$

where

$$T = \sum_{1}^{\infty} \frac{n}{2^n}.$$

Thus

$$S = 2T + 2.$$

Similarly,

$$\begin{aligned} 2T &= \sum_{1}^{\infty} \frac{n}{2^{n-1}} \\ &= \sum_{0}^{\infty} \frac{n+1}{2^n} \\ &= T + 2. \end{aligned}$$

It follows that

$$T = 2, \quad S = 6.$$

2. A stick is broken in random in 2 places (the 2 break-points being chosen independently). What is the probability that the 3 pieces form a triangle?

   Answer: *We may suppose the stick has length 1. Let the 2 breaks occur at distance x and y along the stick. We can represent this case by the point (x, y) in the square $0 \leq x, y \leq 1$. The probability will be given by the area in this square corresponding to breaks which give pieces that can form a triangle.*

   The condition for this is

   $$x < 1/2, \quad y < 1/2, \quad 1 - x - y < 1/2.$$

   These inequalities define a triangle in the square, of area 1/4. Hence the probability is 1/4.

3. For which real numbers $x > 0$ is there a real number $y > x$ such that

   $$x^y = y^x \ ?$$

   Answer: *Taking logs (to base e),*

   $$y \log x = x \log y.$$

   Thus

   $$\frac{\log x}{x} = \frac{\log y}{y}.$$

   Consider the function

   $$f(x) = \frac{\log x}{x}.$$

   Differentiating,

   $$f'(x) = \frac{1 - \log x}{x^2}.$$

   As $x \to 0+$, $f(x) \to -\infty$; and as $x \to \infty$, $f(x) \to 0$. Thus $f(x)$ increases from 0 to e, where it takes the value e^{-1}, and then decreases to 0. Also $f(x) < 0$ for $0 < x < 1$, and $f(x) > 0$ for $x > 1$

   It follows that there is a $y > x$ with $f(y) = f(x)$ if and only if

   $$1 < x < e.$$

4. Show that there are an infinity of natural numbers n such that when the last digit of n is moved to the beginning (as eg $1234 \mapsto 4123$) n is multiplied by 3.

Answer: *Let*

$$\begin{array}{ccccc} a_n & a_{n-1} & \dots & a_1 & a_0 \\ & & & & 3 \\ \hline a_0 & a_n & \dots & a_2 & a_1 \end{array}$$

First solution: it is clear that $a_0 \geq 3$, since it appears as the first digit on the bottom.

Let us try $a_0 = 3$. Then $a_1 = 9$, and our sum starts

$$\begin{array}{ccc} \dots & 9 & 3 \\ & & 3 \\ \hline \dots & & 9 \end{array}$$

But then $a_2 = 7$:

$$\begin{array}{cccc} \dots & 7 & 9 & 3 \\ & & & 3 \\ \hline \dots & & 7 & 9 \end{array}$$

Continuing in this way, we determine $a_3, a_4, \dots$, successively. After a long time we find we have completed a cycle, and are back where we started:

$$\begin{array}{r} 1034482758620689655172413793 \\ 3 \\ \hline 3103448275862068965517241379 \end{array}$$

This number with 28 digits is a solution to our problem; and we see that the cycle could be repeated any number of times to give an infinity of solutions with $2 \times 28, 3 \times 28, \dots$ digits.

Second proof: Let

$$n = 10b + a,$$

where

$$a = a_0, \quad b = 10^{n-1}a_n + 10^{n-2}a_{n-1} + \cdots + a_1.$$

Then

$$3(10b + a) = 10^n a + b.$$

Thus

$$29a = (10^n - 3)b.$$

Hence

$$29 \mid 10^n - 3.$$

On the other hand, if this is true then it is easy to see that we get a solution with

$$a = \frac{10^n - 3}{29}, \quad b = 1.$$

(We will also have solutions with $b = 2$ and $b = 3$.) Thus we have to show that there are an infinity of solutions n of

$$10^n \equiv 3 \bmod 29.$$

This follows by a little group theory, applied to the multiplicative group $(\mathbb{Z}/29)^\times$ formed by the 28 non-zero remainders modulo 29. By Lagrange's Theorem, the order of 10 in this group divides 28, and is thus 2, 4, 7, 14 or 28. But

$$10^2 = 100 \equiv -12 \bmod 29, \quad 10^4 \equiv 12^2 = 144 \equiv 3 \bmod 29.$$

Thus

$$10^{28q+4} \equiv 3 \bmod 29,$$

giving an infinity of solutions to our problem.

5. What is the whole number part of

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{1997}} ?$$

Answer: *I was surprised no-one made a serious effort at this, as the basic idea — to approximate the sum $\sum f(n)$, where $f(x)$ is an increasing or decreasing function, by the integral $\int f(x)\,dx$ — is quite often used. It is the basis for example of the standard derivation of Stirling's approximation to $n!$, which on taking logs reduces to approximating $\sum \log n$.*

In our case we can approximate

$$S = \sum_{n=1}^{N} \frac{1}{\sqrt{n}}$$

by

$$\int \frac{1}{\sqrt{x}}\,dx = [2\sqrt{X}].$$

Since $1/\sqrt{x}$ is decreasing,

$$\int_1^{N+1} \frac{1}{\sqrt{x}}\,dx \le \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \le 1 + \int_1^N \frac{1}{\sqrt{x}}\,dx.$$

But is that good enough? Almost certainly not, since the 2 values differ by almost 1. However, it cannot be out by more than 1. Thus

$$2(\sqrt{(1998)} - 1) < I < 1 + 2(\sqrt{(1997)} - 1).$$

Now

$$45^2 = 81 \times 25 = 2025, \quad 44^2 = 45^2 - 90 + 1 = 1936.$$

Thus

$$\sqrt{(1997)} \approx 44.7$$

and so

$$[S] = 87 \text{ or } 88.$$

There are two ways of improving the estimate. We could start further into the sum, which would bring the bounds together; for example

$$\int_4^{N+1} \frac{1}{\sqrt{x}}\, dx \le I - 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} \le \frac{1}{2} + \int_4^N \frac{1}{\sqrt{x}}\, dx.$$

An alternative way — which we shall follow — is to take the integral

$$\int_{n-1/2}^{n+1/2} f(x)\, dx$$

as an estimate for $f(n)$.

In our case this means taking

$$2\sqrt{(n = 1/2)} - 2\sqrt{(n - 1/2)}$$

as an approximation to $1/\sqrt{n}$. Since the function $f(x) = 1/\sqrt{x}$ is concave, it follows that

$$\begin{aligned} \frac{1}{\sqrt{n}} &< \int_{n-1/2}^{n+1/2} \frac{1}{\sqrt{x}}\, dx \\ &= 2\left(\sqrt{(n + 1/2)} - \sqrt{(n - 1/2)}\right). \end{aligned}$$

It follows that

$$\begin{aligned} S &> 1 + 2(\sqrt{1997.5} - \sqrt{1.5}) \\ &\approx 1 + 2(44.7 - 1.2) \\ &\approx 88. \end{aligned}$$

We conclude that

$$[S] = 88.$$

6. Prove that

$$n(n+1)(n+2) > \left(n + \frac{8}{9}\right)^3$$

for any integer $n \geq 3$.

Answer: *This is the easiest question on the paper, even if it is stated in a slightly misleading way, since the result has nothing to do with integers. In effect it concerns the polynomial*

$$\begin{aligned} f(x) &= x(x+1)(x+2) - \left(x + \frac{8}{9}\right)^3 \\ &= \left(3 - \frac{2^3}{3}\right)x^2 + \left(2 - \frac{2^6}{3^3}\right)x - \frac{2^9}{3^6}. \end{aligned}$$

This is a quadratic, and $f(x) \to \infty$ as $x \to -\infty$ and as $x \to \infty$. It follows that $f(x)$ has a minimum where $f'(x) = 0$, ie where

$$\frac{2}{3}x = \frac{10}{27},$$

that is,

$$x = \frac{5}{9}.$$

It follows that $f(x)$ is increasing for $x \geq 3$; so it is sufficient to show that

$$f(3) > 0,$$

ie

$$3 \cdot 4 \cdot 5 \geq \left(\frac{35}{9}\right)^3,$$

ie

$$3^7 \cdot 4 \geq 7^3 \cdot 5^2,$$

ie

$$8748 \geq 8575.$$

7. Show that the determinant of the 3×3 matrix

$$A = \begin{pmatrix} \sin(x_1 + y_1) & \sin(x_1 + y_2) & \sin(x_1 + y_3) \\ \sin(x_2 + y_1) & \sin(x_2 + y_2) & \sin(x_2 + y_3) \\ \sin(x_3 + y_1) & \sin(x_3 + y_2) & \sin(x_3 + y_3) \end{pmatrix}$$

is zero for all real numbers $x_1, x_2, x_3, y_1, y_2, y_3$.

Answer: *This problem can be solved by computing the determinant* Δ *directly. But the following argument is quicker.*

Consider Δ *as a function of* x_1*. Evidently*

$$\Delta = A \sin x_1 + B \cos x_1,$$

where A, B *are functions of* x_2, x_3, y_1, y_2, y_3*.*

The determinant vanishes if $x_1 = x_2$*, since the first 2 rows are then equal. Similarly it vanishes if* $x_1 = x_3$*. It follows that*

$$A \sin x_2 + B \cos x_2 = 0 = A \sin x_3 + B \cos x_3.$$

This implies that either $A = B = 0$*, in which case* $\Delta = 0$*, or else*

$$\sin x_2 \cos x_3 = \sin x_3 \cos x_2,$$

ie

$$\tan x_2 = \tan x_3,$$

ie

$$x_2 - x_3 = n\pi$$

for some $n \in \mathbb{N}$*. We can write this*

$$x_2 \equiv x_3 \bmod \pi.$$

Similarly, if $\Delta \neq 0$ *then*

$$x_3 \equiv x_1 \bmod \pi, \quad x_1 \equiv x_2 \bmod \pi.$$

It follows that 2 (at least) of x_1, x_2, x_3 *differ by a multiple of* 2π*, say*

$$x_1 \equiv x_2 \bmod 2\pi.$$

But then the first 2 rows of Δ *are equal, and* Δ *vanishes.*

8. Let

$$A(m, n) = \frac{m!(2m + 2n)!}{(2m)!n!(m + n)!}$$

for non-negative integers m and n. Show that

$$A(m, n) = 4A(m, n - 1) + A(m - 1, n)$$

for $m \geq 1,\ n \geq 1$. Hence or otherwise show that $A(m, n)$ is always an integer.

Answer: *If* $m, n \geq 1$,

$$\begin{aligned}
4A(m, n-1) + A(m-1, n) &= 4\frac{m!(2m+2n-2)!}{(2m)!(n-1)!(m+n-1)!} + \frac{(m-1)!(2m+2n-2)!}{(2m-2)!n!(m+n-1)!} \\
&= \frac{(m-1)!(2m+2n-2)!}{(2m)!n!(m+n-1)!}\left(4mn + 2m(2m-1)\right) \\
&= \frac{m!(2m+2n-2)!}{(2m)!n!(m+n-1)!}\left(2(2n+2m-1)\right) \\
&= \frac{m!(2m+2n-1)!}{(2m)!n!(m+n-1)!}2 \\
&= \frac{m!(2m+2n)!}{(2m)!n!(m+n)!} \\
&= A(m, n).
\end{aligned}$$

On the other hand,

$$A(m, 0) = \frac{m!(2m)!}{(2m)!m!} = 1,$$

while

$$A(0, n) = \frac{(2n)!}{n!n!} = C_n^{2n},$$

an integer.

But now it follows by induction on $d = m + n$ *that* $A(m, n)$ *is an integer. This is true when* $d = 0$*, since* $A(0, 0) = 1$ *as we have seen. Now suppose it true for all* m, n *with* $m+n = d$*. Then if* $m+n = d+1$*, the terms on the right in the recursion formula* $A(m, n) = 4A(m, n-1) + A(m-1, n)$ *have already been proved integral, and so* $A(m, n)$ *is also integral.*

9. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a real polynomial of degree $n \geq 2$ such that

$$0 < a_0 < -\sum_{k=1}^{[n/2]} \frac{1}{2k+1} a_{2k}$$

(where $[n/2]$ denotes the integer part of $n/2$). Prove that the equation $P(x) = 0$ has at least one solution in the range $-1 < x < 1$.

Answer: *The conditions can be written*

$$a_0 > 0, \quad a_0 + \frac{1}{3}{}_{-}3 + \frac{1}{5}{}_{-}5 + \cdots < 0.$$

But

$$\int_{-1}^{1} P(x)\,dx = 2\left(a_0 + \frac{1}{3}{}_{-}3 + \frac{1}{5}{}_{-}5 + \cdots\right).$$

Thus

$$f(0) > 0, \quad \int_{-1}^{1} f(x)\,dx < 0.$$

From the second condition, $f(x) < 0$ for some $x \in (-1, 1)$. Thus $f(x)$ changes sign in $(-1, 1)$, and so has a zero there.

10. Suppose $a_1, a_2, a_3, \ldots$ is an infinite sequence of real numbers satisfying $0 < a_n \le 1$ for all n. Let $S_n = a_1 + a_2 + \cdots + a_n$ and $T_n = S_1 + S_2 + \cdots + S_n$. Show that

$$\sum_{n=1}^{\infty} \frac{a_n}{T_n} < \infty.$$

Answer: *The idea is to "bunch together" the a_n's, and treat each bunch as one element.*

More precisely, if $\sum a_n$ converges then the result is immediate, since $T_n \ge a_1$ and so

$$\sum \frac{a_n}{T_n} \le \frac{1}{a_1} \sum a_n.$$

We may assume therefore that

$$\sum a_n = \infty.$$

Now let us divide the a_n's successively into bunches, each with sum ≥ 1 but < 2, say

$$1 \le a_1 + a_2 + \cdots + a_{m_1} < 2, \quad 1 \le a_{m_1+1} + \cdots + a_{m_2} < 2,$$

and so on. Then

$$S_{m_1} \ge 1,\ S_{m_2} \ge 2,\ S_{m_3} \ge 3, \ldots$$

and

$$T_{m_1} \ge 1,\ T_{m_2} \ge 3,\ S_{m_3} \ge 6, \ldots.$$

In general

$$S_{m_r} \ge r, \quad T_{m_r} \ge \frac{r(r+1)}{2}.$$

But now

$$\begin{aligned}\frac{a_{m_r+1}}{T_{m_r+1}} + \cdots + \frac{a_{m_{r+1}}}{T_{m_{r+1}}} &\le \frac{a_{m_r+1} + \cdots + a_{m_{r+1}}}{T_{m_r}} \\ &\le \frac{2}{T_{m_r}}.\end{aligned}$$

Thus

$$\begin{aligned} \sum \frac{a_n}{T_n} &\leq 2\left(\frac{1}{T_{m_1}} + \frac{1}{T_{m_2}} + \cdots\right) \\ &\leq \sum \frac{1}{r(r+1)} < \infty \end{aligned}$$