Irish Intervarsity Mathematics Competition 1994

University College Dublin

1. Find the sum of the series
$$\sum_{n=1}^{\infty} \frac{n}{n^4+4}.$$

2. Prove that there exist infinitely many positive integers n such that $2n+1$ and $3n+1$ are both perfect squares.

3. Let A, C be points in the plane and B the midpoint of $[AC]$. Let S be the circle with centre A and radius $|AB|$ and T the circle with centre C and radius $|AC|$. Suppose S and T intersect in R, R'. Let S', T' be the circles with centres R, R' and radii $AR = AR'$. Suppose S' and T' intersect in A and C'. Prove that C' is the midpoint of $[AB]$.

4. Let $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ be a polynomial with integer coefficients a_i. Suppose that z is an integer and that
$$P\left(P\left(P(P(z))\right)\right) = z.$$
Prove that $P(P(z)) = z$.

5. Let a, b, c $(a < b < c)$ be the lengths of the sides of a triangle opposite the interior angles A, B and C, respectively. Prove that if a^2, b^2, c^2 are in arithmetic progression, then so are $\cot(A), \cot(B), \cot(C)$.

6. Let
$$A = \begin{pmatrix} 1994 & 1993 \\ 1995 & 1994 \end{pmatrix}.$$
Prove that A can be written as the product $X_1 X_2 \cdots X_r$ where $r \geq 1$ and each
$$X_i \in \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\} \quad (i = 1, 2, \ldots, r)$$
in exactly one way.

7. Noah had 8 species of animals to fit into 4 cages of the ark. He planned to put two species in each cage. It turned out that for each species, there were at most three other species with which it could not share a cage. Could Noah have carried out his plan while arranging that each species shares with a compatible species?

8. The function
$$\sum_{n=1}^{\infty} \frac{n x^n}{1 - x^n} \quad (|x| < 1)$$
is expanded as a power-series $\sum_{k=1}^{\infty} a_k x^k$. Prove that $n = 1994$ is the largest even integer with $a_n = n + 1000$.

9. Let a, b, c be positive real numbers. Prove that
$$[(a+b)(b+c)(c+a)]^{1/3} \geq \frac{2}{\sqrt{3}}(ab + bc + ca)^{1/2}.$$

10. Let $n > 1$ be a positive integer. Prove that
$$\sum_{j=1}^{n-1} j \operatorname{cosec}^2\left(\frac{\pi j}{n}\right) = \frac{n(n^2-1)}{6}.$$