Irish Intervarsity Mathematics Competition 1993

University College Galway

Answer all ten questions

1. Evaluate $\sum_{j=2}^{\infty}\left(\sum_{i=2}^{\infty}\frac{1}{i^j}\right)$.

   Answer: *Since the terms are all positive, the order of summation does not matter:*

$$\sum_{j=2}^{\infty}\sum_{i=2}^{\infty}\frac{1}{i^j}=\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{i^j}$$

But

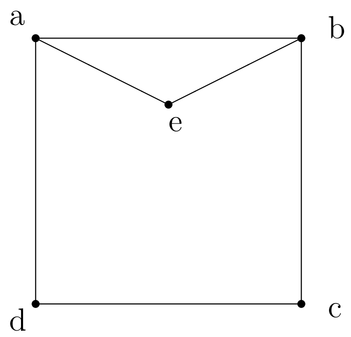
$$\begin{aligned}\sum_{j=2}^{\infty}\frac{1}{i^j} &= \frac{1}{i^2}+\frac{1}{i^3}+\cdots\\ &= \frac{1}{i^2}\left(1-\frac{1}{i}\right)^{-1}\\ &= \frac{1}{i(i-1)}.\end{aligned}$$

Thus

$$\sum_{i=2}^{\infty}\sum_{j=2}^{\infty}\frac{1}{i^j} = \sum_{i=2}^{\infty}\frac{1}{i(i-1)}$$

$$
\begin{aligned}
&= \sum_{i=2}^{\infty}\left(\frac{1}{i-1}-\frac{1}{i}\right) \\
&= 1.
\end{aligned}
$$

2. $abcd$ is a square and e is a point inside the square such that $|ae| = |be|$ and $e\hat{b}a = 15°$. Show that dec is an equilateral triangle.



Answer: *Let the square have side 1. If dec is an equilateral triangle then*

$$|de| = |dc| = |ad|.$$

Thus ade is an isosceles triangle. This gives 2 ways of computing $|ae|$*. Firstly, from the isosceles triangle aeb,*

$$|ae| = \frac{1}{2\cos 15°}.$$

Secondly, from the isosceles triangle ade, since $d\hat{a}e = 75°$*,*

$$|ae| = 2\cos 75° = 2\sin 15°.$$

Hence

$$4\sin 15° \cos 15° = 1.$$

This is true, since $2\sin 15° \cos 15° = \sin 30° = \frac{1}{2}$.

Conversely, working backwards we deduce that the triangle ade is *isosceles, since the perpendicular from d to ae bisects ae; and so* $|de| = |ab| = |dc|$*. Similarly* $|ce| = |dc|$*. Thus the triangle dec is equilateral.*

3. If x, y and n are all natural numbers and $t = x^{4n} + y^{4n} + x^{2n}y^{2n}$ is a prime number, find all possible values of t.

Answer: *Clearly* $x \neq 0,\ y \neq 0$. *We have*

$$\begin{aligned} t &= \left(x^{2n} + y^{2n}\right)^2 - (x^n y^n)^2 \\ &= \left(x^{2n} + y^{2n} - x^n y^n\right)\left(x^{2n} + y^{2n} + x^n y^n\right) \end{aligned}$$

If t *is prime, the first of these factors must be 1, giving*

$$(x^n - y^n)^2 + x^n y^n = 1,$$

which is only possible if

$$x^n - y^n = 0, \quad x^n y^n = 1,$$

in which case

$$\begin{aligned} t &= \left(x^{2n} + y^{2n} + x^n y^n\right) \\ &= (x^n - y^n)^2 + 3x^n y^n \\ &= 3. \end{aligned}$$

4. Construct, with proof, a non–constant arithmetic progression of positive integers which contains no squares, cubes or higher powers of integers.

   Answer: *Suppose* p *is a prime. Consider any arithmetic progression of the form*

   $$p + kp^2 n \quad (n = 0, 1, \ldots)$$

   Each term in the progression is divisible by p*, but not by* p^2*. Hence it cannot be a square or higher power.*

   The simplest sequence of this form, with $p = 2$ *is*

   $$2, 6, 10, 14, \ldots$$

   Each term is divisible by 2, but not by 4.

5. If ${}^nC_r$ is the number of combinations of n objects r at a time, find the value of $\sum_{r=0}^{n} \left(\frac{1}{r+1}\right) {}^nC_r$.

   Answer: *By the binomial theorem,*

   $$\sum_{r=0}^{n} {}^nC_r x^r = (1 + x)^n.$$

Integrating from 0 to 1,

$$\sum_{r=0}^{n} {}^nC_r \int_0^1 x^r\,dx = \int_0^1 (1+x)^n\,dx,$$

giving

$$\begin{aligned}\sum_{r=0}^{n} \left(\frac{1}{r+1}\right)\ {}^nC_r &= \left[\frac{(1+x)^{n+1}}{n+1}\right]_0^1 \\ &= \frac{2^{n+1}-1}{n+1}.\end{aligned}$$

6. If x and y are real numbers, find all solutions of the equation

$$(x^4+1) = (x^2)(2^{1-y^2}).$$

Answer: *[*There was an error in tn earlier version of this: x^2 for x^4 on the left.*] We have to find x, y such that*

$$x^2 + \frac{1}{x^2} = 2^{1-y^2}.$$

Since the arithmetic mean is greater than the geometric,

$$x^2 + \frac{1}{x^2} \geq 2,$$

with equality only if $x^2 = 1$, ie $x = \pm 1$. On the other hand,

$$2^{1-y^2} \leq 2,$$

with equality only if $y = 0$. So the only solutions to the given equation are: $(x, y) = (\pm 1, 0)$.

7. Find a formula for the nth derivative of the function $f(x) = \dfrac{q}{x^2+x^2}$ where a is a constant.

Answer: *There is obviously a misprint in this question.*

8. If abc is an obtuse angled triangle with $b\hat{a}c > \frac{\pi}{2}$ prove that

$$54|ac|^3|ab|^3|\cos b\hat{a}c| \leq |bc|^6$$

and find conditions under which equality holds.

Answer: *By the cosine law,*

$$|bc|^2 = |ab|^2 + |ac|^2 + 2|ab||ac|(-\cos b\hat{a}c).$$

(Note that $\cos b\hat{a}c < 0$*.) Since the arithmetic mean of the 3 terms on the right is less than their geometric mean,*

$$\frac{|ab|^2 + |ac|^2 + 2|ab||ac|(-\cos b\hat{a}c)}{3} \geq \left(2|ab|^3|ac|^3|\cos b\hat{a}c|\right)^{1/3}.$$

Cubing,

$$|bc|^6 \geq 2 \cdot 27|ab|^3|ac|^3|\cos b\hat{a}c|.$$

The arithmetic and geometric means are equal only when all the terms are equal. So there is equality in this case if and only if

$$|ab|^2 = |ac|^2 = 2|ab||ac||\cos b\hat{a}c|;$$

in other words,

$$|ab| = |ac|, \ |\cos b\hat{a}c| = \frac{1}{2}.$$

Thus there is equality if and only if bac is an isosceles triangle with angle $2\pi/3 = 120^\circ$*.*

9. Show that there do *not* exist polynomials $f(x)$ and $g(x)$ such that

$$e^x = \frac{f(x)}{g(x)} \quad \text{for all } x.$$

Answer: *Well,* e^x *increases faster than any polynomial ...*

10. Find the real factors of $x^4 + y^4 + (x+y)^4$.

Answer: *We may observe that if* $y = \omega x$*, where* $\omega = e^{2\pi i/3}$*, then*

$$x + y = (1+\omega)x = -\omega^2 x.$$

Thus

$$\begin{aligned} x^4 + y^4 + (x+y)^4 &= \left(1 + \omega^4 + (-\omega^2)^4\right) x^4 \\ &= (1 + \omega + \omega^2)x^4 \\ &= 0. \end{aligned}$$

Thus $x - \omega y$ *is a factor, and similarly so is* $x - \omega^2$*. Hence one real factor is*

$$(x - \omega y)(x - \omega^2 y) = x^2 + xy + y^2.$$

Dividing

$$x^4 + y^4 + (x+y)^4 = 2\left(x^4 + 2x^3y + 3x^2y^2 + 2xy^3 + y^4\right)$$

by $x^2 + xy + y^2$, *we see that this factor occurs twice:*

$$x^4 + y^4 + (x+y)^4 = 2\left(x^2 + xy + y^2\right)^2.$$