Irish Intervarsity Mathematics Competition 1992

Trinity College Dublin

9.30–12.30 15$^{\text{th}}$ February

Answer all questions.
Calculators permitted.

1. Solve the equation
$$(x-2)(x-3)(x+4)(x+5) = 44.$$

2. Find the greatest value of
$$\frac{x+2}{2x^2+3x+6}.$$

3. Writing numbers to the base 8, show that there are infinitely many numbers which are doubled by reversing their 'digits'.

4. Show that for any positive real numbers a, b with $a > b$,
$$a^n - b^n > n(a-b)(ab)^{(n-1)/2}.$$

5. Describe geometrically the points P, Q in an arbitrary triangle ABC that minimize
   (a) $AP + BP + CP$
   (b) $AQ^2 + BQ^2 + CQ^2$.

6. For m a positive integer, let $k(m)$ denote the largest integer k such that 2^k divides $m!$. Let $c(m)$ denote the number of 1's in the binary representation of m. Show that $k(m) = m - c(m)$.

7. If x_1, x_2, …, x_n are positive numbers and s is their sum, prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leq 1 + s + \frac{s^2}{2!} + \cdots + \frac{s^n}{n!}.$$

8. A table tennis club with 20 members organises 14 singles games (2-player games) one Saturday morning in such a way that each member plays at least once. Show that there must be 6 games involving 12 different players.

9. Let a_1, a_2, … be the sequence of all positive integers with no 9's in their decimal representation. Show that the series

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

converges.

10. In a convex quadrilateral $ABCD$, let E and F be the midpoints of the sides BC and DA (respectively). Show that the sum of the areas of the triangles EDA and FBC is equal to the area of the quadrilateral.