Irish Intervarsity Mathematics Competition 1992

Trinity College Dublin

9.30–12.30 15^{th} February

Answer all questions.
Calculators permitted.

1. Solve the equation
$$(x-2)(x-3)(x+4)(x+5) = 44.$$

Answer: *Let $x-2 = y-3$, ie $y = x+1$. Then the equation becomes*
$$(y-3)(y-4)(y+3)(y+4) = 44.$$
In other words,
$$(y^2-9)(y^2-16) = 44.$$
Let $z = y^2$. We have a quadratic
$$(z-9)(z-16) = 44.$$
This gives
$$z^2 - 25z + 100 = 0.$$
Let $z = 5u$. Then
$$\begin{aligned} 25u^2 - 125u + 100 = 0 &\implies u^2 - 5u + 4 = 0 \\ &\implies (u-4)(u-1) = 0 \\ &\implies u = 1 \text{ or } 4 \\ &\implies z = 5 \text{ or } 20 \\ &\implies y = \pm\sqrt{5} \text{ or } \pm 2\sqrt{5} \\ &\implies x = \pm\sqrt{5} - 1 \text{ or } \pm 2\sqrt{5} - 1. \end{aligned}$$

2. Find the greatest value of

$$\frac{x+2}{2x^2+3x+6}.$$

Answer: *To* maximize

$$f(x) = \frac{x+2}{2x^2+3x+6},$$

it is sufficient to minimize

$$g(x) = \frac{2x^2+3x+6}{x+2}.$$

Setting $x + 2 = y$*,*

$$\begin{aligned} g(x) &= \frac{2(y-2)^2 + 3(y-2) + 6}{y} \\ &= \frac{2y^2 - 5y + 8}{y} \\ &= 2y - 5 + \frac{8}{y} \\ &= 2\left(\sqrt{y} - \frac{2}{\sqrt{y}}\right)^2 + 3. \end{aligned}$$

Thus

$$g_{\min} = 3,$$

the minimum being attained when $y = 2$*. Hence*

$$f_{\max} = \frac{1}{3},$$

the minimum being attained when $x = 0$*.*

3. Writing numbers to the base 8, show that there are infinitely many numbers which are doubled by reversing their 'digits'.

Answer: *Consider 2 'digit' numbers* $x = ab$*, ie*

$$x = 8a + b \qquad (0 \le a, b \le 7).$$

This is doubled on reversal if

$$\begin{aligned} 8b + a = 2(8a + b) &\implies 15a = 6b \\ &\implies 5a = 2b. \end{aligned}$$

This has the solution $a = 2, b = 5$*, ie*

$$2 \cdot 25_8 = 52_8.$$

Now we see that any number of the form

$$2525 \cdots 25_8$$

has the required property.

4. Show that for any positive real numbers a, b with $a > b$,

$$a^n - b^n > n(a-b)(ab)^{(n-1)/2}.$$

Answer: *We have*

$$\frac{a^n - b^n}{a-b} = a^{n-1} + ba^{n-2} + \cdots + b^{n-1}.$$

Thus

$$\frac{a^n - b^n}{n(a-b)} = \frac{1}{n}\left(a^{n-1} + ba^{n-2} + \cdots + b^{n-1}\right).$$

The quantity on the right is the arithmetic mean of $a^{n-1}, ba^{n-2}, \ldots, b^{n-1}$. *Using the result that* Arithmetic Mean $\geq$ Geometric Mean, *with equality only when all terms are equal, we deduce that*

$$\begin{aligned} \frac{a^n - b^n}{n(a-b)} &> \left(a^{n-1} \cdot ba^{n-2} \cdot \cdots \cdot b^{n-1}\right)^{\frac{1}{n}} \\ &= (ab)^{\frac{n-1}{2}}. \end{aligned}$$

5. Describe geometrically the points P, Q in an arbitrary triangle ABC that minimize

   (a) $AP + BP + CP$

   (b) $AQ^2 + BQ^2 + CQ^2$.

Answer: *(a) The point P minimizing $AP + BP + CP$ is the unique point P such that*

$$A\hat{P}B = B\hat{P}C = C\hat{P}A = \frac{2\pi}{3}(= 120^\circ).$$

(To construct this point, draw a circular arc on AB such that $A\hat{P}B = 2\pi/3$ for all points P on the arc; and a similar circular arc on BC such that $B\hat{P}C = 2\pi/3$ for all point on the arc. Then the required point P is the 2nd point—apart from the point B—where these 2 arcs meet.)

Suppose that P minimizes $AP + BP + CP$. Keep CP fixed, allowing P to vary on a circle centred on C. Draw the tangent t to this circle at P.

A small variation dP of P along the tangent t must leave $AP + BP$ unchanged, to the first order in dP. It follows that P must be the point on t minimizing $AP+BP$.

But by the Law of Reflection, this will occur when AP and BP make the same angle θ with t. (To see this, let B' be the reflection of B in t. Then

$$AP + BP = AP + PB'$$

will be minimized when APB' is a straight line.)

But now

$$A\hat{P}C = \theta + \frac{\pi}{2} = B\hat{P}C.$$

Thus at the minimizing point,

$$A\hat{P}C = B\hat{P}C.$$

By the same argument with B in place of C,

$$A\hat{P}B = C\hat{P}B.$$

Hence all 3 angles $A\hat{P}B, B\hat{P}C, C\hat{P}A$ are equal; and since they add up to 2π ($360°$), each must be equal to $2\pi/3$ ($120°$).

(b) The point minimizing $AP^2+BP^2+CP^2$ is the centroid G of ABC. For suppose Q is the minimal point. Let A' be mid-point of BC. Then

$$AQ^2 + BQ^2 = 2A'Q^2 + 2A'B^2.$$

So we have to minimize

$$2A'Q^2 + QA^2.$$

Evidently this will be minimal when AQA' lie in a straight line. Thus Q must lie on the median AA'. Similarly it must lie on the other 2 medians. Hence it lies at the centroid G of ABC.

6. For m a positive integer, let $k(m)$ denote the largest integer k such that 2^k divides $m!$. Let $c(m)$ denote the number of 1's in the binary representation of m. Show that $k(m) = m - c(m)$.

   Answer: *To compute $k(m)$, consider how many of the numbers $1, 2, \ldots, m$ are divisible by 2^i.*

   $\left[\frac{m}{2}\right]$ *numbers are divisible by 2.*
   $\left[\frac{m}{4}\right]$ *numbers are divisible by 4.*
   $\cdots$

   We deduce that

$$k(m) = \left[\frac{m}{2}\right] + \left[\frac{m}{4}\right] + \cdots.$$

   Note that

$$m = \frac{m}{2} + \frac{m}{4} + \cdots.$$

   Consider repeated division of m by 2. At the first step we obtain $[\frac{m}{2}]$, with remainder 1 if m is odd, ie if the last bit of m is 1. At the second step we obtain $[\frac{m}{4}]$, with remainder 1 if the second last bit of m is 1. And so on.

   We conclude that the sum of the remainders is equal to $c(m)$, the number of bits of m equal to 1. Thus

$$k(m) = m - c(m).$$

7. If x_1, x_2, ..., x_n are positive numbers and s is their sum, prove that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \le 1+s+\frac{s^2}{2!}+\cdots+\frac{s^n}{n!}.$$

Answer: *We have*

$$e^x = 1+x+\frac{x^2}{2!}+\cdots.$$

Thus

$$(1+x_1)\cdots(1+x_n) \le e^{x_1}\cdots x^{x_n} = e^s = 1+s+\frac{s^2}{2!}+\cdots.$$

But we can go further. The left hand side only contains terms of degree $\le n$ *in* $x_1,\ldots,x_n$. *Therefore we need only include terms on the right of degree* $\le n$. *But* $s^{n+1}, s^{n+2}, \ldots$ *only contain terms of degree* $> n$, *and can therefore be omitted. Thus*

$$(1+x_1)(1+x_2)\cdots(1+x_n) \le 1+s+\frac{s^2}{2!}+\cdots+\frac{s^n}{n!}.$$

8. A table tennis club with 20 members organises 14 singles games (2-player games) one Saturday morning in such a way that each member plays at least once. Show that there must be 6 games involving 12 different players.

Answer: *Call the players* $1,\ldots,20$. *Let player* i *play* $1+n_i$ *games. Then the total number of games is*

$$\frac{1}{2}\sum(1+n_i) = 10+\frac{1}{2}\sum n_i.$$

But we know that there are 14 games. Hence

$$\sum n_i = 8.$$

Now let us go through the 20 players. For each player with $n_i > 0$, *delete* n_i *of his games. This leaves at least*

$$\frac{1}{2}(20-8) = 6 \text{ games.}$$

By construction, no player plays in more than 1 of these 6 games.

9. Let a_1, a_2, ... be the sequence of all positive integers with no 9's in their decimal representation. Show that the series

$$\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots$$

converges.

Answer: *Consider numbers with exactly* k *digits. Each such number is*

$$\ge 10\cdots 0 = 10^{k-1}.$$

There are $10^k - 10^{k-1}$ k-digit numbers, of which $9^k - 9^{k-1}$ do not contain the digit 9. Thus the k-digit numbers not containing 9 contribute

$$\begin{aligned} &\leq \left(9^k - 9^{k-1}\right) 10^{-(k-1)} \\ &\leq 10 \left(\frac{9}{10}\right)^k \end{aligned}$$

to the sum. Hence

$$\sum \frac{1}{a_i} \leq 10 \sum \left(\frac{9}{10}\right)^k = 100.$$

In particular the series converges.

10. In a convex quadrilateral $ABCD$, let E and F be the midpoints of the sides BC and DA (respectively). Show that the sum of the areas of the triangles EDA and FBC is equal to the area of the quadrilateral.

Answer: *Writing $|XYZ|$ for the area of the triangle XYZ,*

$$|EDA| = \frac{1}{2}|DA| \times p,$$

where p is the perpendicular distance from E to DA. But

$$p = \frac{1}{2}(p_1 + p_2).$$

where p_1, p_2 are the perpendicular distances from B, C to DA, respectively. Hence

$$\begin{aligned} |EDA| &= \frac{1}{4}|DA| \times p_1 + \frac{1}{4}|DA| \times p_2 \\ &= \frac{1}{2}\left(|BDA| + |CDA|\right). \end{aligned}$$

Similarly

$$|FBC| = \frac{1}{2}\left(|ABC| + |DBC|\right).$$

Hence

$$\begin{aligned} |EDA| + |FBC| &= \frac{1}{2}\left(|BDA| + |DBC| + |CDA| + |ABC|\right) \\ &= \frac{1}{2}\left(|ABCD| + |ABCD|\right) \\ &= |ABCD| \end{aligned}$$