Royal Life Ireland

Irish Intervarsity Mathematics Competition

Trinity College Dublin 1991

9.30–12.30 February 9

Answer all questions.
Calculators permitted.

1. The prime factorisations of $r+1$ positive integers ($r \geq 1$) together involve only r primes. Prove that there is a subset of these integers whose product is a perfect square.

2. Consider a polynomial $p(x) = x^n + nx^{n-1} + a_2x^{n-2} + \cdots + a_n$ which has real roots $r_1, r_2, \ldots, r_n$. If
$$r_1^{16} + r_2^{16} + \cdots + r_n^{16} = n,$$
find all the roots.

3. An n-inch cube (n a positive integer) is painted on all sides and then cut into 1-inch cubes. If the number of small cubes with one painted side is the same as the number with two painted sides, what could n have been?

4. How many ways are there of painting the 6 faces of a cube in 6 different colours, if two colourings are considered the same when one can be obtained from the other by rotating the cube?

5. How many positive integers $x \leq 1991$ are such that 7 divides $2^x - x^2$?

6. How many ways can 1,000,000 be expressed as a product of 3 positive integers? Factorisations different only in order are considered to be the same.

7. Prove that 2^n can begin with any sequence of digits.

8. Imagine a point P inside a square $ABCD$. If $|PA| = 5$, $|PB| = 3$ and $|PC| = 7$, what is the side of the square?

9. Let $f(x)$ be a function such that $f(1) = 1$ and, for $x \geq 1$

$$f'(x) = \frac{1}{x^2 + f^2(x)}.$$

   Prove that

$$\lim_{x \to \infty} f(x)$$

   exists and is less than $1 + \pi/4$.

10. Prove that the number of odd binomial coefficients in each row of Pascal's triangle is a power of 2. [In Pascal's triangle

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \\ & & & \vdots & & & \end{array}$$

   each entry is the sum of the entries directly above it.]