Irish Intervarsity Mathematics Competition

University College Cork 1990

1. If x, y and z are real numbers solve the equations

$$\begin{aligned} x+y+z &= \sqrt[3]{2} \\ x^2+y^2+z^2 &= \sqrt[3]{4} \\ x^3+y^3+z^3 &= 2 \end{aligned}$$

2. A hand of 13 cards is dealt from a pack of 52 cards containing 13 spades, 13 diamonds, 13 clubs and 13 hearts. If the hand has 3 clubs, 4 spades, 2 diamonds and 4 hearts, it is siad to have the distribution type $4-4-3-2$, while if it has 6 hearts and 7 spades it is said to have distribution type $7-6$. How many distribution types are there? List them all.

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

4. Find the minimum value of the real function

$$f(x) = x^x, \qquad \text{for } x > 0.$$

5. Prove that

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$$

6. A ladder of constant length $2L$ extends from a horizontal floor to a wall which is inclined at $85°$ to the horizontal. If the ladder slips while remaining in contact with both the floor and the wall, find the locus of the mid-point of the ladder.

7. Prove that 3, 5 and 7 are the only 3 consecutive odd numbers all of which are prime.

8. If x, y and z are positive integers, all greater than 1, find *any* solution of the equation
$$x^x y^y = z^z.$$

9. If a, b, c are the lengths of the side of a triangle prove that
$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca),$$
and in each case find the necessary and sufficient condition.

10. Let S be the set of all natural numbers, which can be written in the form $x^3 + y^3 + z^3 - 3xyz$ for some $x, y, z \in \mathbf{N}$, where $\mathbf{N}$ is the set of natural numbers. if $a \in S$ and $b \in S$ prove that $ab \in S$.