



# Irish Intervarsity Mathematics Competition

University College Cork 1990

1. If  $x$ ,  $y$  and  $z$  are real numbers solve the equations

$$\begin{aligned}x + y + z &= \sqrt[3]{2} \\x^2 + y^2 + z^2 &= \sqrt[3]{4} \\x^3 + y^3 + z^3 &= 2\end{aligned}$$

**Answer:** *We can simplify the equations by setting*

$$x = \sqrt[3]{2}X, \quad y = \sqrt[3]{2}Y, \quad z = \sqrt[3]{2}Z.$$

*The equations become*

$$\begin{aligned}X + Y + Z &= 1 \\X^2 + Y^2 + Z^2 &= 1 \\X^3 + Y^3 + Z^3 &= 1\end{aligned}$$

*From the power-sums on the left-hand side of these equations we can compute the symmetric products*

$$c_1 = X + Y + Z, \quad c_2 = YZ + ZX + XY, \quad c_3 = XYZ.$$

*The solutions to the equations are then given by*

$$\{X, Y, Z\} = \{\alpha, \beta, \gamma\},$$

where  $\alpha, \beta, \gamma$  are the roots of a cubic

$$t^3 - c_1t^2 + c_2t - c_3 = 0.$$

But it is clear from the equations that one solution is

$$(X, Y, Z) = (1, 0, 0).$$

It follows that there are just 3 solutions:

$$(X, Y, Z) = (1, 0, 0), (0, 1, 0), (0, 0, 1),$$

2. A hand of 13 cards is dealt from a pack of 52 cards containing 13 spades, 13 diamonds, 13 clubs and 13 hearts. If the hand has 3 clubs, 4 spades, 2 diamonds and 4 hearts, it is said to have the distribution type  $4 - 4 - 3 - 2$ , while if it has 6 hearts and 7 spades it is said to have distribution type  $7 - 6$ . How many distribution types are there? List them all.

**Answer:** We are asked, in effect, for the number  $p_4(13)$  of partitions of 13 into 4 or fewer parts. It is well-known that there is no simple formula for  $p_r(n)$ , although there are various recurrence formulae, expressing  $p_r(n)$  in terms of partitions of smaller numbers.

However, we are asked to list the partitions, so there seems little point in looking for a nice method. The partitions are:

13,  
 12 - 1,  
 11 - 2, 11 - 1 - 1,  
 10 - 3, 10 - 2 - 1, 10 - 1 - 1 - 1,  
 9 - 4, 9 - 3 - 1, 9 - 2 - 2, 9 - 2 - 1 - 1,  
 8 - 5, 8 - 4 - 1, 8 - 3 - 2, 8 - 3 - 1 - 1, 8 - 2 - 2 - 1,  
 7 - 6, 7 - 5 - 1, 7 - 4 - 2, 7 - 4 - 1 - 1, 7 - 3 - 3, 7 - 3 - 2 - 1, 7 - 2 - 2 - 2,  
 6 - 6 - 1, 6 - 5 - 2, 6 - 5 - 1 - 1, 6 - 4 - 3, 6 - 4 - 2 - 1, 6 - 3 - 3 - 1, 6 - 3 - 2 - 2,  
 5 - 5 - 3, 5 - 5 - 2 - 1, 5 - 4 - 4, 5 - 4 - 3 - 1, 5 - 4 - 2 - 2, 5 - 3 - 3 - 2,  
 4 - 4 - 4 - 1, 4 - 4 - 3 - 2, 4 - 3 - 3 - 3.

Thus

$$p_4(13) = 39.$$

3. Evaluate

$$\sum_{n=1}^{\infty} \frac{n^3}{3^n}.$$

**Answer:** *We have*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{x^n}{3^n} &= \left(1 - \frac{x}{3}\right)^{-1} \\ &= \frac{3}{3-x}. \end{aligned}$$

*Differentiating,*

$$\sum_{n=1}^{\infty} \frac{nx^{n-1}}{3^n} = \frac{3}{(3-x)^2}$$

*Multiplying by  $x$ ,*

$$\sum_{n=1}^{\infty} \frac{nx^n}{3^n} = \frac{3x}{(3-x)^2}$$

*Differentiating again,*

$$\sum_{n=1}^{\infty} \frac{n^2x^{n-1}}{3^n} = \frac{3}{(3-x)^2} + \frac{6x}{(3-x)^3}.$$

*Multiplying by  $x$ ,*

$$\sum_{n=1}^{\infty} \frac{n^2x^n}{3^n} = \frac{3x}{(3-x)^2} + \frac{6x^2}{(3-x)^3}.$$

*Differentiating a third time,*

$$\sum_{n=1}^{\infty} \frac{n^3x^{n-1}}{3^n} = \frac{3}{(3-x)^2} + \frac{18x}{(3-x)^3} + \frac{18x^2}{(3-x)^4}.$$

*Substituting  $x = 1$ ,*

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^3}{3^n} &= \frac{3}{4} + \frac{18}{8} + \frac{18}{16} \\ &= \frac{33}{8}. \end{aligned}$$

4. Find the minimum value of the real function

$$f(x) = x^x, \quad \text{for } x > 0.$$

**Answer:** We have

$$\log f(x) = x \log x.$$

Hence

$$\frac{f'(x)}{f(x)} = \log x + 1.$$

Thus

$$f'(x) = 0 \iff \log x = -1 \iff x = e^{-1}.$$

As  $x \rightarrow 0+$ ,  $\log f(x) \rightarrow 0$  and so  $f(x) \rightarrow 1$ . As  $x \rightarrow \infty$ ,  $\log f(x) \rightarrow \infty$  and so  $f(x) \rightarrow \infty$ . Thus the minimum value is

$$f(e^{-1}) = e^{-e^{-1}} = \frac{1}{e^{1/e}}.$$

5. Prove that

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$$

**Answer:** We have

$$\tan 7\theta = \frac{P(\tan \theta)}{Q(\tan \theta)},$$

where  $P(t)$  and  $Q(t)$  are polynomials of degree  $\leq 7$  in  $t$ . Now

$$\tan 7\theta = 0 \iff \theta = \frac{n\pi}{7}$$

for some integer  $n$ . It follows that the roots of  $P(t) = 0$  are the 7 numbers

$$\tan \frac{n\pi}{7}, \quad (n = 0, 1, \dots, 6).$$

In fact, since

$$\tan \frac{(7-n)\pi}{7} = -\tan \frac{n\pi}{7},$$

the roots are

$$0, \pm \tan \frac{\pi}{7}, \pm \tan \frac{2\pi}{7}, \pm \tan \frac{3\pi}{7}.$$

Hence

$$P(t) = x(at^6 + bt^4 + ct^2 + d).$$

Thus

$$\tan^2\left(\frac{\pi}{7}\right), \tan^2\left(\frac{2\pi}{7}\right), \tan^2\left(\frac{3\pi}{7}\right)$$

are the roots of

$$ax^3 + bx^2 + cx + d.$$

In particular

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = -\frac{b}{a}.$$

It only remains to compute the polynomial  $P(t)$ . Writing  $\tan \theta = t$ , we have

$$\begin{aligned}\tan 2\theta &= \frac{2t}{1-t^2}, \\ \tan 3\theta &= \frac{t + \frac{2t}{1-t^2}}{1 - \frac{2t^2}{1-t^2}} \\ &= \frac{3t - t^3}{1 - 3t^2}, \\ \tan 4\theta &= \frac{2\frac{2t}{1-t^2}}{1 - \frac{4t^2}{(1-t^2)^2}} \\ &= \frac{4t(1-t^2)}{1 - 6t^2 + t^4}, \\ \tan 7\theta &= \frac{\frac{3t-t^3}{1-3t^2} + \frac{4t(1-t^2)}{1-6t^2+t^4}}{1 - \frac{3t-t^3}{1-3t^2} \frac{4t(1-t^2)}{1-6t^2+t^4}} \\ &= \frac{P(t)}{Q(t)},\end{aligned}$$

where

$$\begin{aligned}P(t) &= (3t - t^3)(1 - 6t^2 + t^4) + 4t(1 - t^2)(1 - 3t^2) \\ &= -t[t^6 - 21t^4 + 34t^2 - 7].\end{aligned}$$

Hence

$$\tan^2\left(\frac{\pi}{7}\right) + \tan^2\left(\frac{2\pi}{7}\right) + \tan^2\left(\frac{3\pi}{7}\right) = 21.$$

6. A ladder of constant length  $2L$  extends from a horizontal floor to a wall which is inclined at  $85^\circ$  to the horizontal. If the ladder slips while remaining in contact with both the floor and the wall, find the locus of the mid-point of the ladder.

**Answer:** Taking cartesian coordinates centred on the point where the floor and wall meet, the coordinate of the point where the ladder meets the wall is

$$(Y \sin 5^\circ, Y \cos 5^\circ)$$

for some  $Y$ . If the point where the ladder meets the floor is  $(X, 0)$  then

$$(X - \sin 5^\circ Y)^2 + (\cos 5^\circ Y)^2 = 4L^2.$$

In other words

$$X^2 + Y^2 - 2 \sin 5^\circ XY = 4L^2.$$

The mid-point of the ladder has coordinates

$$(x, y) = \left( \frac{X + \sin 5^\circ Y}{2}, \frac{\cos 5^\circ Y}{2} \right).$$

Thus

$$Y = 2 \sec 5^\circ y, \quad X = 2x - 2 \tan 5^\circ y.$$

Thus the point  $(x, y)$  lies on the locus

$$(x - \tan 5^\circ y)^2 + \sec^2 5^\circ y^2 - 2 \sin 5^\circ (x - \tan 5^\circ y) \sec 5^\circ y = L^2.$$

Thus the locus of the mid-point of the ladder is the ellipse

$$x^2 + (\tan^2 5^\circ + \sec^2 5^\circ)y^2 - 2 \tan 5^\circ xy = L^2;$$

or rather, that part of the ellipse between the wall and the floor.

7. Prove that 3, 5 and 7 are the only 3 consecutive odd numbers all of which are prime.

**Answer:** One of the 3 numbers  $n, n + 2, n + 4$  is divisible by 3. So if all 3 are prime, one of the numbers must be 3. Thus the 3 numbers must be 3, 5, 7.

8. If  $x, y$  and  $z$  are positive integers, all greater than 1, find any solution of the equation

$$x^x y^y = z^z.$$

**Answer:** [Solution by James Ward, of UCG] Any solution must have

$$z < x + y,$$

since

$$(x + y)^{x+y} > x^x y^y$$

(as one of the terms in the binomial expansion of the left-hand side will be a multiple of the right-hand side).

Suppose

$$d = \text{hcf}(x, y).$$

Then

$$d^{x+y} | z^z \implies d | z,$$

since  $z < x + y$ . Let

$$x = dX, \quad y = dY, \quad z = dZ.$$

Then (taking the  $d$ th root on each side)

$$d^e X^X Y^Y = Z^Z,$$

where  $e = X + Y - Z$ .

In particular

$$X^X Y^Y | Z^Z.$$

Moreover, if we can find  $X, Y, Z$  satisfying this relation and also satisfying

$$X + Y - Z = 1$$

then  $dX, dY, dZ$  will satisfy the original equation, with

$$d = \frac{Z^Z}{X^X Y^Y}.$$

Suppose

$$u - v = 1.$$

Squaring,

$$u^2 + v^2 - 2uv = 1.$$

Let us therefore take

$$X = u^2, \quad Y = v^2, \quad Z = 2uv.$$

Then

$$\begin{aligned} \frac{Z^Z}{X^X Y^Y} &= 2^{2uv} u^{2uv-2u^2} v^{2uv-2v^2} \\ &= \frac{2^{2uv} v^{2v}}{u^{2u}}. \end{aligned}$$

We can make this to be an integer—as we want—by taking  $u = 2^a$ . In this case 2 occurs to the power  $2uv$  on the top, and  $2au$  on the bottom. Thus the condition will be satisfied if

$$v = 2^a - 1 \geq a.$$

This holds for all  $a$ . However, we only get a non-trivial solution if  $a \geq 2$ .

So we have the family of solutions

$$X = 2^{2a}, Y = (2^a - 1)^2, Z = 2^{a+1}(2^a - 1)$$

for  $a = 2, 3, \dots$

Taking  $a = 2$  this gives

$$X = 2^4 = 16, Y = 3^2 = 9, Z = 2^3 \cdot 3 = 24.$$

In this case  $d = 2^8 3^6$ , and so

$$x = 2^{12} 3^6, y = 2^8 3^8, z = 2^{11} 3^7.$$

9. If  $a, b, c$  are the lengths of the side of a triangle prove that

$$3(ab + bc + ca) \leq (a + b + c)^2 \leq 4(ab + bc + ca),$$

and in each case find the necessary and sufficient condition.

**Answer:** Taking the inequality on the left first, we have

$$\begin{aligned} (a + b + c)^2 - 3(ab + bc + ca) &= a^2 + b^2 + c^2 - (ab + bc + ca) \\ &= \frac{1}{2}(a - b)^2 + \frac{1}{2}(b - c)^2 + \frac{1}{2}(c - a)^2. \end{aligned}$$

Thus the inequality holds, with equality if and only if  $a = b = c$ , ie the triangle is equiangular.

The inequality on the right can be written

$$2(ab + bc + ca) - (a^2 + b^2 + c^2) \geq 0.$$

Now

$$ab + bc = b(a + c) \geq b^2,$$



since  $a + c \geq b$ . Similarly

$$bc + ca \geq c^2, \quad ca + ab \geq a^2.$$

Adding the 3 inequalities,

$$2(ab + bc + ca) \geq a^2 + b^2 + c^2.$$

Now  $a = b + c$  only if the 3 vertices  $A, B, C$  lie on a line with  $A$  between  $B$  and  $C$ . Thus the 3 inequalities cannot all be equalities simultaneously unless the 3 vertices coincide, ie  $a = b = c = 0$ .

10. Let  $S$  be the set of all natural numbers, which can be written in the form  $x^3 + y^3 + z^3 - 3xyz$  for some  $x, y, z \in \mathbf{N}$ , where  $\mathbf{N}$  is the set of natural numbers. if  $a \in S$  and  $b \in S$  prove that  $ab \in S$ .

**Answer:** We have

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z),$$

where  $\omega = e^{2\pi i/3}$ .

Now if  $a, b, c$  are integers then

$$a + b\omega + c\omega^2 = 0 \iff a = b = c \iff a + b\omega^2 + c\omega = 0.$$

Suppose

$$m = x^3 + y^3 + z^3 - 3xyz, \quad n = X^3 + Y^3 + Z^3 - 3XYZ.$$

We can certainly satisfy the equation

$$u + \omega v + \omega^2 w = (x + \omega y + \omega^2 z)(X + \omega Y + \omega^2 Z)$$

by taking

$$u = xX + yZ + zY, \quad v = xY + yX + zZ, \quad w = xZ + yY + zX.$$

Moreover, by the argument above this will also imply that

$$u + \omega^2 v + \omega w = (x + \omega^2 y + \omega z)(X + \omega^2 Y + \omega Z)$$

And it is easy to see that

$$u + v + w = (x + y + z)(X + Y + Z).$$

Multiplying these factors together,

$$mn = u^3 + v^3 + w^3 - 3uvw.$$

Thus  $mn \in S$ .