Irish Intervarsity Mathematics Competition 2007

Trinity College Dublin

Time allowed: 3 hours

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. Does the number 2007^n end with the digits 2007 for any $n > 1$?

   Answer: *The answer is Yes. The multiplicative group* $(\mathbb{Z}/10000)^\times$ *has order*

$$\begin{aligned}\phi(10000) &= \phi(2^4 5^4)\\ &= \phi(2^4)\phi(5^4)\\ &= 2^3 \cdot 4 \cdot 5^3\\ &= 2000.\end{aligned}$$

   Hence, by Lagrange's Theorem,

$$2007^{2000} \equiv 1 \bmod 10000,$$

   and so

$$2007^n \equiv 2007 \bmod 1000$$

   if

$$n = 2000m + 1.$$

2. Does the number 2007^n begin with the digits 2007 for any $n > 1$?

   Answer: *The answer is Yes. The result follows if we can find* n *such that*

$$2007^n = 10000000000\ldots,$$

   ie

$$2007^n = 10^m(1 + \epsilon),$$

where $0 < \epsilon < 0.0000000001$*; for then*

$$2007^{n+1} = 2007\ldots,$$

Taking logarithms to base 10, and setting $\alpha = log_{10}2007$*,*

$$n\alpha = m + \log_{10}(1+\epsilon),$$

which will hold if

$$0 \le [n\alpha] \le \epsilon,$$

We can find such an n *for any real number* α*. (This is Kronecker's Theorem.) For if we consider the remainders* $[n\alpha]$*, two must differ by less than* ϵ *(by the Pigeon Hole Principle), say*

$$0 \le [r\alpha] - [s\alpha] \le \epsilon,$$

and so

$$0 \le [n\alpha] \le \epsilon,$$

for $n = r - s$*.*

3. Three ants A,B,C start at the vertices of an equilateral triangle. Ant A pursues B, B pursues C, and C pursues A (each moving always in the direction of its target).

   If the sides of the triangle are 1 metre in length, and the ants move at 1mm/sec, how long does it take them to meet at the centroid of the triangle?

   Answer: *The 3 ants will always lie at the vertices of an equilateral triangle* $A(t)B(t)C(t)$*, with the same centroid* O*.*

   The angle
   $$\angle OA(t)B(t) = \pi/6.$$
   Hence the component of the velocity of ant A *towards* O *is*
   $$v = 1 \cdot \cos(\pi/6) = \sqrt{3}/2.$$
   Initially
   $$AO = \frac{2}{3}\cdot\frac{\sqrt{3}}{2}\cdot 1000.$$
   Hence the time taken for the ants to reach O *is*
   $$T = \frac{2000}{3} \textit{ seconds.}$$

4. Is the circle the only convex figure with the property that every inscribed equilateral triangle is of the same size?

   Answer:

5. A triangle ABC is given.

    (a) What point P minimizes $AP + BP + CP$?

    (b) What point P minimizes $AP^2 + BP^2 + CP^2$?

    Answer:

6. Does there exist a map $f : \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) such that

$$f(f(x)) = x^2$$

for all $x \in \mathbb{Z}$?

Answer: *The answer is Yes.*

Note that if we set

$$s(x) = x^2$$

then

$$f^2 = s \implies sf = fs,$$

ie

$$f(x^2) = f(x)^2.$$

In particular,

$$f(0)^2 = f(0),\ f(1)^2 = f(1).$$

It follows that either

$$f(0) = 0, f(1) = 1 \text{ or } f(0) = 1, f(1) = 0.$$

More generally, if

$$f(a) = b$$

then

$$f(b) = a^2,\ f(a^2) = b^2,\ f(b^2) = a^3, \ldots.$$

Let us divide the integers $n > 1$ into chains

$$C(r) = \{r, r^2, r^4, r^8, \ldots\}.$$

Thus $\mathbb{N} \setminus \{0, 1\}$ is partitioned into

$$C(2), C(3), C(5), C(6), \ldots,$$

with a chain $C(r)$ starting with each non-square r.

Divide the chains into pairs $(C(2), C(3))$, $(C(5), C(6)), \ldots$ Suppose $C(r), C(s)$ is one pair. Then we can set

$$f(r^i) = s^i,\ f(s^i) = r^{i+1}.$$

We can extend the definition to $\mathbb{Z}$ by setting

$$f(-n) = f(n).$$

7. Show that every rational number $x \in (0,1)$ can be represented uniquely in the form

$$x = \frac{a_1}{1!} + \frac{a_2}{2!} + ... + \frac{a_k}{k!},$$

where $a_1, \ldots, a_k$ are integers with $0 \le a_i < i$ for $1 \le i \le k$.

Answer: *Notice that* $a_1 = 0$*, since* $0 \le 1_1 < 1$*.*

Let

$$a_2 = [2x],$$

and let

$$x_2 = 2x - a_2.$$

Then

$$0 \le a_2 < 2,$$

and

$$0 \le x_2 < 1.$$

Let

$$a_3 = [3x_2],$$

and let

$$x_3 = 3x_2 - a_3.$$

Then

$$0 \le a_3 < 3,$$

and

$$0 \le x_3 < 1.$$

Continuing in this way, the process will end if and when

$$x_{k+1} = 0.$$

It is easy to see that this will happen when

$$k!x \in \mathbb{N},$$

as must be true for some k*.*

8. What is the greatest number of parts into which the plane can be divided by n straight lines?

Answer:

9. Three points A, B, C are chosen at random on the circumference of a circle. What is the probability that the centre of the circle lies inside ABC?

Answer: *Suppose the angle subtended by* AB *at the centre is* θ*. Then* C *must lie on an arc subtending the same angle at the centre. (In fact*

C must lie in the reflection of the arc AB in the centre. The probability of this is $\theta/2\pi$. Thus the probability that the centre lies in ABC is

$$\begin{aligned} p &= \frac{1}{\pi}\int_0^\pi \frac{\theta}{2\pi}\,d\theta \\ &= \frac{1}{2\pi^2}\left[\frac{\theta^2}{2}\right]_0^\pi \\ &= \frac{1}{4}. \end{aligned}$$

10. A circular hole of diameter 1 is drilled through the centre of a sphere of radius 1. What is the volume of the drilled sphere?

    Answer: *We calculate the volume removed. We can regard this as made up of cylindrical shells of radius r and thickness dr, as r varies from 0 to $1/2$. The height of the cylindrical shell is h, where*

    $$x^2 + h^2 = 1,$$

    ie

    $$h = \sqrt{1 - x^2},$$

    and its volume is

    $$2\pi x h.$$

    Hence the volume removed is

    $$V = 2\pi \int_0^{1/2} x\sqrt{1-x^2}dx.$$

    Setting $x = \sin\theta$ this gives

    $$V = 2\pi \int_0^{\pi/3} \sin\theta\cos\theta\cos\theta\;d\theta.$$

    Now

    $$\begin{aligned} 2\sin\theta\cos^2\theta &= \sin\theta(1-\cos 2\theta) \\ &= \sin\theta - \sin\theta\cos 2\theta \\ &= \sin\theta - \frac{1}{2}(\sin 3\theta - \sin\theta) \\ &= \frac{1}{2}(3\sin\theta - \sin 3\theta). \end{aligned}$$

    Hence

    $$\begin{aligned} V &= \frac{\pi}{2}[-3\cos\theta - \cos 3\theta/3]_0^{\pi/3} \\ &= \frac{\pi}{6}(3 - 3\sqrt{3}2 + 2/3)// \qquad = \frac{\pi}{18}(11 - 3\sqrt{3}). \end{aligned}$$

    Hence the volume remaining is

    $$\frac{4\pi}{-}V = \frac{\pi}{18}(61 + 3\sqrt{3}).$$