Irish Intervarsity Mathematics Competition 2007

Trinity College Dublin

11.00–14.00 Saturday 14th April 2007

Answer as many questions as you can; all carry the same mark. Give reasons in all cases.
Tables and calculators are not allowed.

1. Does the number 2007^n end with the digits 2007 for any $n > 1$?

2. Does the number 2007^n begin with the digits 2007 for any $n > 1$?

3. Three ants A,B,C start at the vertices of an equilateral triangle. Ant A pursues B, B pursues C, and C pursues A (each moving always in the direction of its target).

   If the sides of the triangle are 1 metre in length, and the ants move at 1mm/sec, how long does it take them to meet at the centroid of the triangle?

4. Is the circle the only convex figure with the property that every inscribed equilateral triangle is of the same size?

5. A triangle ABC is given.

   (a) What point P minimizes $AP + BP + CP$?
   (b) What point P minimizes $AP^2 + BP^2 + CP^2$?

6. Does there exist a map $f : \mathbb{Z} \to \mathbb{Z}$ (where $\mathbb{Z}$ is the set of integers) such that
   $$f(f(x)) = x^2$$
   for all $x \in \mathbb{Z}$?

More questions overleaf!

7. Suppose x, y are positive integers. Show that if

$$\frac{x^2 + y^2}{xy + 1}$$

is an integer then it is a perfect square.

8. Show that every rational number $x \in (0, 1)$ can be represented uniquely in the form

$$x = \frac{a_1}{1!} + \frac{a_2}{2!} + ... + \frac{a_k}{k!},$$

where $a_1, \ldots, a_k$ are integers with $0 \leq a_i < i$ for $1 \leq i \leq k$.

9. What is the greatest number of parts into which the plane can be divided by n straight lines?

10. Three points A, B, C are chosen at random on the circumference of a circle. What is the probability that the centre of the circle lies inside ABC?

11. For which real numbers x does the sequence

$$x,\ \sin x,\ \sin(\sin x),\ \sin(\sin(\sin x)), \ldots$$

converge?

12. A circular hole of diameter 1 is drilled through the centre of a sphere of radius 1. What is the surface area of the drilled sphere?