Irish Intervarsity Mathematics Competition 2003

University College Dublin

Time allowed: Three hours

1. Let
$$f(x) = x^3 + Ax^2 + Bx + C,$$
where A, B, C are integers. Suppose the roots of $f(x) = 0$ (in the field of complex numbers) are α, β, γ. Prove that if
$$|\alpha| = |\beta| = |\gamma| = 1$$
then
$$f(x) \mid (x^{12} - 1)^3.$$

   Answer: *One or three of the roots must be real. But if $\alpha \in \mathbb{R}$ and $|\alpha| = 1$ then $\alpha = \pm 1$.*

   If the three roots are ± 1 then the result follows, since ± 1 are roots of $x^{12} - 1$.

   So we may assume that one root, say α, is ± 1, and the other two are complex conjugates $e^{\pm\theta}$. Since $\alpha + \beta + \gamma = -A$ is an integer, so is $\beta + \gamma = 2\cos\theta$. Thus either $\beta + \gamma = 0$, in which case $\beta, \gamma = \pm i$; or else $\beta + \gamma = \pm 1$, in which case $\beta, \gamma = \omega, \omega^2$ or $-\omega, -\omega^2$, where $\omega = e^{2\pi i/3}$.

   Since $\pm i, \pm\omega, \pm\omega^2$ are all roots of $x^{12} - 1$, the result follows.

2. Let n be a positive integer. Prove that when written in decimal form (in base 10),
$$\left(\sqrt{17} + 4\right)^{2n+1}$$
has at least n zeroes following the decimal point.

   Answer: *Let*
$$x = \sqrt{17} + 4,\ y = \sqrt{17} - 4.$$

Then

$$xy = 1;$$

while

$$x^{2n+1} - y^{2n+1} \in \mathbb{Z},$$

since the terms involving odd powers of $\sqrt{17}$ *cancel out.*

It follows that the part of x^{2n+1} *after the decimal point is* y^{2n+1}. *This gives the result, since* $x > 8$ *and so*

$$y^{2n+1} < 64^{-n} < 10^{-n}.$$

3. Find all integers n for which

$$n^4 - 16n^3 + 86n^2 - 176n + 169$$

is the square of an integer.

Answer: *Let the given expression be* $f(n)$, *and let*

$$g(n, c) = n^2 - 8n + c,$$

for integers c. *Then*

$$g(n, c)^2 = n^4 - 16n^3 + (64 + 2c)n^2 - 16cn + c^2.$$

Thus

$$\begin{aligned} f(n) &= g(n, 11)^2 + 48, \\ &= g(n, 12)^2 - 2n^2 + 16n + 25, \\ &= g(n, 13)^2 - 4n^2 + 32n. \end{aligned}$$

It follows that if $f(n) = m^2$ *then* $m > g(n, 11)$. *But* $m \leq g(n, 13) = g(n, 11) + 2$ *unless*

$$4n^2 \leq 32n,$$

ie

$$0 \leq n \leq 8.$$

If $n = 0$ or $n = 8$ then $f(n) = g(n, 13)^2$. So we need only consider $1 \le n \le 7$. We have

$$g(n, 11) = (n - 4)^2 - 5.$$

Thus

$$\begin{aligned}
f(1) &= g(1, 11)^2 + 48 = 4^2 + 48 = 64 = 8^2,\\
f(2) &= g(2, 11)^2 + 48 = (-1)^2 + 48 = 49 = 7^2,\\
f(3) &= g(3, 11)^2 + 48 = (-4)^2 + 48 = 64 = 8^2,\\
f(4) &= g(4, 11)^2 + 48 = (-5)^2 + 48 = 73,\\
f(5) &= g(5, 11)^2 + 48 = (-4)^2 + 48 = 64 = 8^2,\\
f(6) &= g(6, 11)^2 + 48 = (-1)^2 + 48 = 49 = 7^2,\\
f(7) &= g(7, 11)^2 + 48 = 4^2 + 48 = 64 = 8^2.
\end{aligned}$$

Finally, if $f(n)$ is a square for n outside the range $[0, 8]$ then $f(n) = g(n, 12)^2$, in which case

$$2n^2 - 16n + 25 = 0,$$

which is impossible since the first two terms are even while the last is odd.

We conclude that $f(n)$ is a square if and only if $n \in \{0, 1, 2, 3, 5, 6, 7, 8\}$.

4. Consider the sequence

$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, 5, 5, 5, 5, 5, \ldots$$

in which each positive integer k is repeated k times. Prove that its n^{th} term is

$$\left[\frac{1 + \sqrt{8n - 7}}{2}\right],$$

where $[x]$ denotes the greatest integer not exceeding x.

Answer: *Let the nth number in the sequence be a_n.*

The first n for which $a_n = k$ is

$$n = 1 + 2 + \cdots + (k - 1) + 1 = \frac{k(k - 1)}{2} + 1 = \frac{k^2 - k + 2}{2}.$$

The function

$$f(x) = \frac{x^2 - x + 2}{2}$$

is monotone increasing for $x \geq 1$. *Thus*

$$n = f(x)$$

for a unique $x = x(n) \geq 1$; *and* $a_n = k$ *if*

$$k \leq x(n) < k + 1,$$

ie

$$a_n = [x(n)].$$

But $x(n)$ *is the solution of*

$$x^2 - x + 2 = 2n.$$

Thus

$$\begin{aligned} a_n &= \left[\frac{1 + \sqrt{1 - 4(2 - 2n)}}{2} \right] \\ &= \left[\frac{1 + \sqrt{8n - 7}}{2} \right]. \end{aligned}$$

5. Let ABC be an acute angled triangle and a, b, c the lengths of the sides BC, CA, AB, respectively. Let P be a point inside ABC, and let x, y, z be the lengths PA, PB, PC, respectively. Prove that

$$(x + y + z)^2 \geq \frac{a^2 + b^2 + c^2}{2}.$$

Answer: *We have*

$$x + y > c,\ y + z > b,\ z + x > a.$$

Thus

$$\begin{aligned} (y + z)^2 + (z + x)^2 + (x + y)^2 &= 2(x^2 + y^2 + z^2 + yz + zx + xy) \\ &> a^2 + b^2 + c^2. \end{aligned}$$

Hence

$$\begin{aligned} 2(x + y + z)^2 &= 2(x^2 + y^2 + z^2 + 2(yz + zx + xy)) \\ &> a^2 + b^2 + c^2, \end{aligned}$$

ie

$$(x + y + z)^2 > (a^2 + b^2 + c^2)/2.$$

6. Let $ABCD$ be a convex quadrilateral with the lengths $AB = AC$, $AD = CD$ and angles $B\hat{A}C = 20°$, $A\hat{D}C = 100°$. Prove that the lengths $AB = BC + CD$.

Answer: *Since the triangle ABC is isosceles,*

$$A\hat{B}C = A\hat{C}B = 80°.$$

Similarly, since the triangle DAC is isosceles,

$$D\hat{A}C = D\hat{C}A = 40°.$$

From the triangle ABC,

$$\frac{BC}{\sin 20} = \frac{AB}{\sin 80}.$$

Thus

$$BC = \frac{\sin 20}{\sin 80} AB.$$

Similarly, from the triangle ACD,

$$\frac{AC}{\sin 100} = \frac{CD}{\sin 40}.$$

Thus

$$CD = \frac{\sin 40}{\sin 100} AC = \frac{sin 40}{sin 80} AB.$$

Accordingly, we have to show that

$$\frac{\sin 20}{\sin 80} + \frac{\sin 40}{\sin 80} = 1,$$

ie

$$\sin 20 + \sin 40 = \sin 80.$$

But

$$\begin{aligned}\sin 20 + \sin 40 &= \sin(30 - 10) + \sin(30 + 10) \\ &= 2 \sin 30 \cos 10 \\ &= \cos 10 \\ &= \sin 80,\end{aligned}$$

as required.

7. Let S be a set of 30 positive integers less than 100. Prove that there exists a nonempty subset T of S such that the product of the elements of T is the square of an integer.

Answer: *If we take the numbers $\{1, 2, \ldots, 100\}$ modulo squares we obtain an abelian group A, in which eg $\bar{3} \cdot \bar{6} = \bar{2}$, The elements of A are all of order 2, and A is generated by the elements $\bar{p}$ defined by primes p. There are 25 primes $p \le 100$, so*

$$A = C_2^{25}.$$

We can regard A as a 25-dimensional vector space over the 2-element field $\mathbb{F}_2$. If $s_1, \ldots, s_r \in S$ then

$$\bar{s_1} + \cdots + \bar{s_r} = 0 \iff s_1 \cdots s_r \text{ is a square.}$$

The 30 elements

$$\{\bar{s} : s \in S\}$$

must be linearly dependent in the vector space A, ie there is some relation

$$\bar{s_1} + \cdots + \bar{s_r} = 0$$

But then the product

$$s_1 \cdots s_r$$

is a square.

8. Let

$$f(x) = x^5 + ax^4 + bx^3 + cx^2 + dx + e,$$

where a, b, c, d, e are integers, and suppose that $f(x) = 0$ has no integer roots. Suppose also that $f(x) = 0$ has roots α, β (in the field of complex numbers) with $\alpha + \beta$ an integer. Show that $\alpha\beta$ is an integer.

Answer: *Suppose*

$$\alpha + \beta = n.$$

Consider the factorisation of $f(x)$ into irreducible polynomials over the rationals $\mathbb{Q}$. We know that any such factorisation is in fact a factorisation into monic polynomials over $\mathbb{Z}$. Since $f(x)$ has no integral root it cannot have a factor of degree 1. Thus either $f(x)$ is irreducible, or else it factorises into 2 irreducible polynomials, of degrees 2 and 3.

Suppose first that $f(x)$ is irreducible. Then $\beta = n - \alpha$ is a root of $f(n - x)$, as well as of $f(x)$. It follows that

$$f(n - x) = -f(x).$$

Thus from the coefficients of x^4,

$$5n + a = -a,$$

ie

$$5n = 2a.$$

In particular, n is even.

The roots of $f(x)$ must divide into pairs $\{\theta, n-\theta\}$ with at least one root satisfying $\theta = n - \theta$. But that is impossible, since $f(x)$ has no integral root. It follows that $f(x)$ cannot be irreducible.

Thus $f(x)$ factorises, say

$$f(x) = g(x)h(x),$$

where α is a root of $g(x)$.

As before, $\beta = n - \alpha$ is a root of $g(n - x)$, as well as of $f(x)$. Thus $g(n-x)$ is (to within a factor ± 1) either $g(x)$ or $h(x)$. Since $\deg g(x) \neq \deg h(x)$,

$$g(n - x) = \pm g(x).$$

Suppose first that $g(x)$ is cubic, say

$$g(x) = x^3 + Ax^2 + Bx + C.$$

If the third root is γ then

$$\alpha + \beta + \gamma = -A \implies \gamma = -(A + n),$$

giving an integral root of $f(x)$, contrary to assumption.

Hence $g(x)$ is quadratic, say

$$g(x) = x^2 + Bx + C.$$

Then $\alpha\beta = C$ is integral, as required.

9. Let x be a real number with $0 < x < 1$. Let $\{a_n\}$ be a sequence of positive real numbers. Prove that the inequality

$$1 + xa_n \geq x^2 a_{n-1}$$

holds for infinitely many positive integers n.

Answer: *Suppose to the contrary that*

$$a_n < xa_{n-1} - \frac{1}{x}$$

for all sufficiently large n, say $n \geq N$.

This implies in particular that a_n is decreasing for $n \geq N$. Hence a_n converges to a limit $\ell \geq 0$, satisfying

$$\ell \leq x\ell - \frac{1}{x}.$$

But this implies that

$$(1-x)\ell \leq -\frac{1}{x} < 0 \implies \ell < 0,$$

which is impossible since $a_n \geq 0$.

10. Find the least positive integer n for which

$$m^n - 1 \text{ is divisible by } 10^{2003}$$

for all integers m with greatest common divisor $\gcd(m, 10) = 1$.

Answer: *We have to find n such that*

$$m^n \equiv 1 \bmod 2^{2003}$$

and

$$m^n \equiv 1 \bmod 5^{2003}.$$

The group $(\mathbb{Z}/2^n)^\times$, ie the group of odd numbers modulo 2^n, contains

$$\phi(2^n) = 2^{n-1}$$

elements. Thus the order of any odd number in this group is 2^r for some r.

It is not hard to show that if $n \geq 2$,

$$(\mathbb{Z}/2^n)^\times \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2^{n-2}).$$

Thus every odd number has order dividing 2^{n-2}, and some odd numbers have this order.

The group $(\mathbb{Z}/5^n)^\times$*, ie the group of numbers coprime to 5 modulo* 5^n*, contains*

$$\phi(5^n) = 4 \cdot 5^{n-1}$$

elements. is 2^r *for some* r*. Again, it is not hard to see that this group is cyclic. Thus every number coprime to 5 has order dividing* $4 \cdot 5^{n-1}$*, and some such numbers have this order.*

It follows that the least number n *such that all* m *coprime to 10 satisfy*

$$m^n \equiv 1 \bmod 10^{2003}$$

is

$$\begin{aligned} n &= \mathrm{lcm}(2^{2000}, 4 \cdot 5^{2001}) \\ &= 2^{2000} 5^{2001} \\ &= 5 \cdot 10^{2000}. \end{aligned}$$