1. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that $A \cap B$ is finite for all $A \neq B$ from this family?

2. Does there exist an infinite uncountable family of subsets of $\mathbb{N}$ such that for all $A \neq B$ from this family either $A \subset B$ or $B \subset A$?

3. Show that a monotonously increasing function is continuous outside some at most countable (that is, finite or countable) set.

4. Show that $n! > \frac{n^n}{e^n}$.

5. Show that the sequences $a_n = \sin(n^2)$ and $b_n = \sin(4^n)$ are not convergent.

6. Let $a_1 = 1/2$, $a_{n+1} = a_n - a_n^2$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite nonzero limit, and compute that limit.

7. Let $a_1 = 1$, $a_{n+1} = \sin(a_n)$. Find a real number c for which the sequence $b_n = n^c a_n$ has a finite nonzero limit, and compute that limit.

8. Let $a_1 = c$, $a_{n+1} = a_n + \frac{a_n^2}{n^2}$. Show that if $0 < c < 1$, then the sequence a_n is bounded from above.

9. Assume that for a sequence of real numbers a_n we have $\lim_{n\to\infty}(a_n + a_{n^2}) = 0$. Does that imply that $\lim_{n\to\infty} a_n = 0$?

10. Let $f(x)$ be a continunous function which assumes both positive and negative values. Show that there exists an arithmetic series a, b, c with $a < b < c$ such that $f(a) + f(b) + f(c) = 0$.

11. Show that the equation $x^{\lfloor x \rfloor} = 9/2$ has no positive rational solutions.

12. Show that the system of equations

$$\begin{cases} z_1 + 2z_2 + \ldots + nz_n = 0, \\ z_1^2 + 2z_2^2 + \ldots + nz_n^2 = 0, \\ \qquad\qquad \ldots \\ z_1^n + 2z_2^n + \ldots + nz_n^n = 0. \end{cases}$$

has the only complex solution $z_1 = \ldots = z_n = 0$.

13. Find all $a > 0$ for which the inequality $a^x \geqslant ax$ holds for all positive x.

14. Show that for all $a < b < c$ there exists a plane section of the surface $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which is a circle, and find the maximal possible radius for such a circle.

15. Assume that for a polynomial $P(x)$ with integer coefficients every value at an integer point is a perfect square. Show that $P(x) = Q(x)^2$ for some polynomial $Q(x)$.

16. Show that for any fixed choice of off-diagonal entries in a square matrix it is possible to assign to each place on the diagonal 0 or 1 in such a way that the resulting matrix has a non-zero determinant.

17. Show that for any choice of five vectors of a Euclidean space V it is possible to pick two of them so that the length of their sum is less than or equal to the length of the sum of three remaining vectors.

18. How many of complex roots of the equation $z^8+z^3+1=0$ have both their imaginary part and their real part positive?

19. For a set of 2011 coins, it is known that if we remove an arbitrary coin from it, the remaining ones can be divided into two set of 1005 coins of the same total weight. Show that all coins have the same weight.

20. Let $S = x_1 + x_2 + \ldots + x_n$. Show that

$$(1+x_1)(1+x_2)\cdots(1+x_n) \leqslant 1 + S + \frac{S^2}{2} + \ldots + \frac{S^n}{n!}.$$

21. Show that if $\cos a = b$ and $\cos b = a$ then $a = b$.

22. Show that for every a and b there exists a point $x \in [-\pi/2, \pi/2]$ for which

$$|\sin x - ax - b| \geqslant 1/8.$$

23. Show that it is impossible to place two triangles, each of area greater than 1, in a circle of radius 1 without overlaps.

24. Show that for every two polynomials $f(t)$ and $g(t)$ there exists a non-zero polynomial $R(x, y)$ in two variables such that $R(f(t), g(t)) = 0$.

25. Compute the integral $\int_0^{\pi/2}(\cos^2(\cos x) + \sin^2(\sin x))\, dx$.

26. The equation $b + ax = \sin x$ has n solutions. How many solutions can the equation $b - ax = \sin x$ have?

27. Find all integer solutions to the equation $x^2 + y^2 + z^2 = 2xyz$.

28. Let $P(x) = x(x-1)(x-2)\cdots(x-100)$. Show that the solution set to the inequality $P'(x)/P(x) > 1$ is a finite union of intervals, and compute the sum of lengths of those intervals.

29. Let a_k be the number of decimal digits of 7^k. Compute the limit of the sequence $b_k = a_k/k$.

30. Find the distance between the graphs $y = e^{2010x}$ and $y = \frac{1}{2010}\ln x$.

31. Show that there exists a real number α for which the fractional part of α^n is between $1/3$ and $2/3$ for all positive integers n.

32. In a group of 1981 people, every person knows at least 45 other people. Show that it is possible to pick four people from that group and sit them at the four sides of a square table so that every one of them sits next to people he or she knows.

33. For two square matrices A and B of the same size, it is known that whenever $Ax = 0$ for some vector x, we also have $Bx = 0$. Show that $B = CA$ for some matrix C.

34. Let y_n be an increasing sequence with $\lim y_n = +\infty$. Show that if, for some sequence x_n, the sequence $a_n = \frac{x_n - x_{n-1}}{y_n - y_{n-1}}$ is convergent, then the sequence $b_n = \frac{x_n}{y_n}$ converges to the same limit.

35. (**a**) Let $f(x)$ be a polynomial in one variable with real coefficients. Show that if $f(x) > 0$ for all real x, then $\inf_{x \in \mathbb{R}} f(x) > 0$.

(**b**) Is a similar statement true for polynomials in two variables?

36. Compute the determinant of the matrix A with entries a_{ij} if

(**a**) $a_{ij} = x_i^{j-1}$

(**b**) $a_{ij} = \frac{1}{x_i + y_j}$

(**c**) $a_{ij} = \gcd(i, j)$

37. Let A be a skew-symmetric matrix of even size. Show that if we add the same number to all entries of A, then the determinant of the resulting matrix is equal to the determinant of A.

38. Show that for rectangular matrices A and B such that the products AB and BA are defined, we have $\det(I - AB) = \det(I - BA)$.

39. Vectors $e_1, \dots, e_k$ of an n-dimensional Euclidean space V satisfy $(e_i, e_j) < 0$ for all $i \neq j$. What is the maximal possible value of k?

40. Let V be the vector space of all $n \times n$-matrices, and assume that $f\colon V \to V$ is an invertible linear operator such that $f(XY) = f(X)f(Y)$ for all X, Y. Show that $f(X) = AXA^{-1}$ for some matrix A.