Exercise 11

Gaussian Integers

In exercises 1-5, factorise the given gaussian integer into (gaussian) primes.

** 1. $3 + 5i$

** 2. $5 + 3i$

** 3. $5 + 12$

** 4. $11 + 2i$

** 5. $15 - 4i$

In exercises 6-10, determine the gcd of the two given gaussian integers.

** 6. $3 + 2i, 2 + 3i$

** 7. $12, 9 - 3i$

** 8. $5 - i, 3 + 2i$

** 9. $4, 10$

** 10. $13 + 2i, 7 - 11i$

In exercises 11-15, either express the given number as a sum of two squares, or else show that this is not possible.

** 11. 33

** 12. 45

** 13. 98

** 14. 99

** 15. 128

*** 16. Find a formula expressing

$$(x^2 + y^2 + z^2 + t^2)(X^2 + Y^2 + Z^2 + T^2)$$

as a sum of 4 squares.

*** 17. Show that every prime p can be expressed as a sum of 4 squares.

** 18. Deduce from the last 2 exercises that every $n \in \mathbb{N}$ can be expressed as a sum of 4 squares.

** 19. Show that if $n \equiv 7 \bmod 8$ then n cannot be expressed as a sum of 3 squares.

*** 20. Show that if $n = 4^e(8m + 7)$ then n cannot be expressed as a sum of 3 squares.

*** 21. Suppose $p \equiv 1 \bmod 4$ is prime. If $p = m^2 + n^2$, find $u, v \in \mathbb{N}$ such that

$$2p = u^2 + v^2,$$

and show that this representation of $2p$ as a sum of 2 squares is unique.

**** 22. Show that if p is a prime such that

$$2p = n^2 + 1$$

then p is the sum of the squares of two consecutive integers.

*** 23. Show that if the prime $p = m^2 + n^2$ and $p \equiv \pm 1 \bmod 10$ then

$$5 \mid xy.$$

*** 24. Find the smallest $n \in \mathbb{N}$ such that $n, n+1, n+2$ are each a sum of 2 squares, but none is a perfect square.

**** 25. Show that there are arbitrarily long gaps between successive integers expressible as a sum of 2 squares.