Exercise 4

In Exercises 1–16 determine all solutions of the given congruence.

* 1. $3x \equiv 1 \bmod 23$

* 2. $7x \equiv 1 \bmod 47$

** 3. $5x \equiv 2 \bmod 210$

** 4. $6x \equiv 7 \bmod 25$

** 5. $8x \equiv 5 \bmod 31$

** 6. $8x \equiv 12 \bmod 32$

** 7. $12x \equiv 6 \bmod 21$

** 8. $2x \equiv 2 \bmod 16$

** 9. $20x \equiv 8 \bmod 24$

*** 10. $7x \equiv -3 \bmod 2009$

** 11. $x^2 \equiv 1 \bmod 12$

** 12. $x^2 \equiv -1 \bmod 15$

** 13. $x^2 + x + 1 \equiv 0 \bmod 3$

** 14. $x^2 - 2x + 3 \equiv 0 \bmod 5$

** 15. $x^2 - 2 \equiv 0 \bmod 7$

*** 16. $x^4 + 2x^2 + x - 2 \equiv 0 \bmod 7$

* 17. What is the order of 10 in the additive group $\mathbb{Z}/(24)$?

** 18. Determine the orders of the elements $7, 11, 21$ in the multiplicative group $(\mathbb{Z}/36)^\times$.

** 19. What is the order of the group $(\mathbb{Z}/36)^\times$?

*** 20. Is the group $(\mathbb{Z}/36)^\times$ cyclic?

*** 21. Is Christmas equally likely to take place on any day of the week?

**** 22. Given integers $x_1, x_2, \ldots, x_{11}$, show that there exists a finite sequence $a_1, ..., a_{11}$ of numbers from $\{-1, 0, 1\}$ such that the sum $a_1x_1 + ... + a_{11}x_{11}$ is divisible by 2009.

*** 23. Construct the field containing 4 elements.

**** 24. Show that there is no field containing 6 elements.

*** 25. Determine the orders of all the elements in $\mathbb{F}_{11}^\times$?

** 26. What is the order of the multiplicative group $\mathbb{F}_q^\times$?

*** 27. How many elements are there of order 4 in $\mathbb{F}_{17}^\times$?

*** 28. Prove that there is a multiple of 2009 which ends with the digits 000001.

In Exercises 21–25 determine the additive order of the given element.

* 29. 3 mod 5

* 30. 3 mod 6

* 31. 2 mod 7

* 32. -13 mod 14

** 33. 100000 mod 123456

In Exercises 26–30 determine the multiplicative order of the given element.

* 34. 3 mod 5

* 35. 7 mod 12

** 36. 2 mod 31

** 37. -2 mod 31

*** 38. 2 mod 3^5

In Exercises 31–35 determine the multiplicative inverse of the given element.

* 39. 3 mod 5

* 40. 3 mod 13

* 41. 2 mod 111

** 42. 137 mod 253

In Exercises 36–40 determine the order of the given multiplicative group, and list its elements.

* 43. $(\mathbb{Z}/2)^\times$

* 44. $(\mathbb{Z}/6)^\times$

* 45. $(\mathbb{Z}/8)^\times$

* 46. $(\mathbb{Z}/12)^\times$

* 47. $(\mathbb{Z}/15)^\times$

* 48. Determine $\phi(45)$

* 49. Determine $\phi(3^n)$

* 50. Determine all positive integers n with $\phi(n) = n - 1$.

** 51. Determine all positive integers n with $\phi(n) = n - 2$.

** 52. What is the smallest value of $\phi(n)/n$?

*** 53. Show that there is a field containing 4 elements.

*** 54. Show that there is no field containing 6 elements.