Course 3413 — Group Representations
Sample Paper II

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Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. What is a *group representation*?

   What is meant by saying that a representation is *simple*?

   What is meant by saying that a representation is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Answer:

   (a) *A representation α of a group G in a vector space V is a homomorphism*

   $$\alpha : G \to \mathrm{GL}(V).$$

   (b) *The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if*

   $$g \in G, u \in U \implies gu \in U.)$$

(c) *The representation α of G in V is said to be semisimple if it can be expressed as a sum of simple representations:*

$$\alpha = \sigma_1 + \cdots + \sigma_m.$$

This is equivalent to the condition that each stable subspace $U \subset V$ has a stable complement W:

$$V = U \oplus W.$$

(d) *Suppose α is a representation of the finite group G in the vector space V. Let*

$$P(u, v)$$

be a positive-definite hermitian form on V. Define the hermitian form Q on V by

$$Q(u, v) = \frac{1}{\|G\|} \sum_{g \in G} H(gu, gv).$$

Then Q is positive-definite (as a sum of positive-definite forms). Moreover Q is invariant under G, ie

$$Q(gu, gv) = Q(u, v)$$

for all $g \in G, u, v \in V$. For

$$\begin{aligned} Q(hu, hv) &= \frac{1}{\|G\|} \sum_{g \in G} H(ghu, ghv) \\ &= \frac{1}{|G|} \sum_{g \in G} H(gu, gv) \\ &= Q(u, v), \end{aligned}$$

since gh runs over G as g does.

Now suppose U is a stable subspace of V. Then

$$U^{\perp} = \{v \in V : Q(u, v) = 0 \forall u \in U\}$$

is a stable complement to U.

2. Show that all simple representations of an abelian group are of degree 1.

   Determine from first principles all simple representations of $D(6)$.

   Answer:

(a) *Suppose α is a simple representation of the abelian group G in V. Suppose $g \in G$. Let λ be an eigenvalue of g, and let $E = E_\lambda$ be the corresponding eigenspace. We claim that E is stable under G. For suppose $h \in G$. Then*

$$e \in E \implies g(he) = h(ge) = \lambda he \implies he \in E.$$

Since α is simple, it follows that $E = V$, ie $gv = \lambda v$ for all v, or $g = \lambda I$.

Since this is true for all $g \in G$, it follows that every subspace of V is stable under G. Since α is simple, this implies that $\dim V = 1$, ie α is of degree 1.

(b) *We have*

$$D_6 = \langle t, s : s^6 = t^2 = 1, st = ts^5 \rangle.$$

Let us first suppose α is a 1-dimensional representations of D_6. ie a homomorphism

$$\alpha : D_6 \to \mathbb{C}^*.$$

Suppose

$$\alpha(s) = \lambda, \alpha(t) = \mu.$$

Then

$$\lambda^6 = \mu^2 = 1, \lambda\mu = \mu\lambda^5.$$

The last relation gives

$$\lambda^4 = 1.$$

Hence

$$\lambda^2 = 1, \ \mu^2 = 1.$$

Thus there are just 4 1-dimensional representations given by

$$s \mapsto \pm 1, t \mapsto \pm 1.$$

Now suppose α is a simple representation of D_6 in the vector space V over $\mathbb{C}$, where $\dim V \geq 2$. Let $e \in V$ be an eigenvector of s:

$$se = \lambda e;$$

and let

$$f = te.$$

Then

$$sf = ste = ts^5 e = \lambda^5 te = \lambda^5 f.$$

It follows that the subspace

$$\langle e, f \rangle \subset V$$

is stable under D_6*, since*

$$se = \lambda e, sf = \lambda^5 f, te = f, tf = t^2 e = e.$$

Since V *by definition is simple, it follows that*

$$V = \langle e, f \rangle.$$

Since $s^6 = 1$ *we have* $\lambda^6 = 1$*, ie* $\lambda = \pm 1, \pm\omega, \pm\omega^2$ *(where* $\omega = e^{2\pi i/3}$*).*
It also follows from the argument above that if λ *is an eigenvalue of* s *then so is* $\lambda^5 = 1/\lambda$*.*
If $\lambda = 1$ *then* s *would have eigenvalues* $1, 1$ *(since* $1^5 = 1$*). But we know that* s *(ie* $\alpha(s)$*) is diagonalisable. It follows that* $s = I$*. Similarly if* $\lambda = -1$ *then* s *has eigenvalues* $-1, -1$ *and so* $s = -I$*. In either of these cases* s *will be diagonal with respect to any basis. Since we can always diagonalise* t*, we can diagonalise* s, t *simultaneously. But in that case the representation would not be simple; since the 1-dimensional space* $\langle e \rangle$ *would be stable under* D_6*.*
Thus we are left with the cases $\lambda = \pm\omega, \pm\omega^2$*. If* $\lambda = \omega^2$ *then on swapping* e *and* f *we would have* $\lambda = \omega$*; and similarly if* $\lambda = -\omega^2$ *then on swapping* e *and* f *we would have* $\lambda = -\omega$*.*
So we have just two 2-dimensional representation (up to equivalence):

$$s \mapsto \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \; t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and

$$s \mapsto \begin{pmatrix} -\omega & 0 \\ 0 & -\omega^2 \end{pmatrix}, \; t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We note that these representations are not equivalent, since in the first case

$$\chi(s) = \operatorname{tr}(\alpha(s)) = \omega + \omega^2 = -1,$$

while in the second case

$$\chi(s) = \operatorname{tr}(\alpha(s)) = -\omega - \omega^2 = 1.$$

3. Determine all groups of order 30.

   Answer: *This is an exercise in Sylow's Theorem and semi-direct products.*

   Suppose $\#G = 30$.

   If G is abelian then we know from the Structure Theorem for Finitely Generated Abelian Groups that

$$G = C_2 \times C_3 \times C_5 = C_{30}.$$

   Three other fairly obvious cases are:

$$G = S_3 \times C_5,$$
$$G = C_3 \times D_5,$$
$$G = D_{15}.$$

   Lemma 1. *C_{15} is the only group of order 15.*

   Proof. By Sylow's Theorem, the number $n(5)$ of Sylow 5-subgroups satisfies

$$n(5) \equiv 1 \bmod 5.$$

   If two such subgroups C_5 have an element $g \neq e$ in common then they are identical (both being generated by g). Hence the Sylow 5-subgroups contain $4n(5) + 1$ elements altogether.

   It follows that $n(5) = 1$, ie

$$C_5 \triangleleft G.$$

   Now let C_3 be a Sylow 3-subgroup. Then C_3 acts on C_5 by

$$(g, x) \mapsto gxg^{-1}.$$

   This defines a homomorphism

$$\alpha : C_3 \to \text{Aut}(C_5) = C_4.$$

   (Thus G is the semi-direct product $C_5 \ltimes_\alpha C_3$.)

   Since there are no elements of order 3 in C_4, α is trivial, and so H is the direct product

$$H = C_5 \times C_3 = C_{15}.$$

$\square$

Lemma 2. *G must contain a subgroup H of order 15.*

Proof. Consider the number $n(5)$ of Sylow 5-subgroups in G. As above, the Sylow 5-subgroups contain $4n(5)+1$ elements in total. Since $n(5) \equiv 1 \bmod 5$ it follows that

$$n(5) = 1 \text{ or } 6.$$

If $n(5) = 1$ then

$$C_5 \triangleleft G.$$

It follows that if C_3 is a Sylow 3-subgroup of G then

$$C_5 C_3 = C_3 C_5$$

is a subgroup of G of order 15. (Note that in any group G, if N, H are subgroups with $N \triangleleft G$ then $NH = \{nh : n \in N, h \in H\}$ is a subgroup of G; and if $N \cap H = \{e\}$ then this subgroup contains $\#N \, \#H$ elements.)

On the other hand, if $n(5) = 6$ then there are 24 elements of order 5 in G, leaving just 6 elements.

But there are $2n(3)$ elements of order 3, and $n(3) \equiv 1 \bmod 3$. It follows that $n(3) = 1$, ie

$$C_3 \triangleleft G.$$

It follows that if C_5 is a Sylow 5-subgroup of G then

$$C_3 C_5 = C_5 C_3$$

is a subgroup of G of order 15. □

Thus we have

$$C_{15} \triangleleft G.$$

It follows that if C_2 is a Sylow-2 subgroup of G then C_2 acts on C_{15} and

$$G = C_{15} \rtimes_\alpha C_2,$$

where

$$\alpha : C_2 \to \mathrm{Aut}(C_{15})$$

is a homomorphism.

Now

$$\begin{aligned}\mathrm{Aut}(C_{15}) &= \mathrm{Aut}(C_3 \times C_5)\\ &= \mathrm{Aut}(C_3) \times \mathrm{Aut}(C_5)\\ &= C_2 \times C_4.\end{aligned}$$

(For any automorphism must send elements of order 3 into elements of order 1 or 3, and similarly for elements of order 5.)

If $C_2 = \{e, g\}$ then g must map into an automorphism of order 1 or 2.

If g maps into the trivial automorphism then the semi-direct product is direct, and

$$G = C_{15} \times C_2 = C_{30}.$$

There are 3 elements of order 2 in $C_2 \times C_4$, namely

$$(1 \bmod 2, 1),\ (1, 2 \bmod 4),\ (1 \bmod 2, 2 \bmod 4).$$

In the first case C_2 acts trivially on C_5, and

$$G = (C_3 \ltimes_\beta C_2) \times C_5.$$

But there is only one non-trivial homomorphism

$$\beta : C_2 \to \mathrm{Aut}(C_3) = C_2,$$

so there is just one non-abelian group of order 6, namely $S_3 = D_3$; and we get just 2 groups in this case,

$$G = C_5 \times C_6 = C_{30} \text{ and } C_5 \times S_3,$$

which we have already noted.

Similarly in the second case C_2 acts trivially on C_3, and

$$G = (C_5 \ltimes_\beta C_2) \times C_3.$$

But there is only one non-trivial homomorphism

$$\beta : C_2 \to \mathrm{Aut}(C_5) = C_4$$

(since C_4 has just one element of order 2, namely $2 \bmod 4$); so there is just one non-abelian group of order 10, namely D_5; and we get just 2 groups in this case,

$$G = C_3 \times C_{10} = C_{30} \text{ and } C_3 \times D_5,$$

which we have already seen.

Finally, the third case gives us

$$G = D_{15}.$$

This follows since it is the last case, and we have not met D_{15} before.

But we can show this directly. If $C_2 = \{e, g\}$ and x, y are elements of order 3 and 5 in C_15 then

$$gxg{-}1 = x^{-1} \textit{ and } gyg{-}1 = y^{-1}.$$

(For these are the two automorphisms of C_3 and C_5 of order 2.) Hence

$$g(xy)g^{-1} = x^{-1}y^{-1} = (xy)^{-1}.$$

(Note that x, y are in the abelian group C_{15}.) Thus we have the standard presentation of D_{15}:

$$D_{15} = \langle g, s : g^2 = s^{15} = 1, gsg^{-1} = s^{-1} \rangle.$$

We conclude that the only groups of order 15 are the 4 groups we listed at the beginning:

$$C_{15}, S_3 \times C_5, C_3 \times D_5 \textit{ and } D_{15}.$$

4. Prove that the number of simple representations of a finite group G is equal to the number of conjugacy classes in G.

   Show that if the finite group G has simple representations $\sigma_1, \ldots, \sigma_s$ then

   $$\deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|.$$

   Determine the degrees of the simple representations of S_6.

   Answer:

   (a)

5. Determine the conjugacy classes in A_5, and draw up the character table of this group.

   Answer:

   (a)

   Lemma 3. *The even class $C = \langle g \rangle$ in S_n splits in A_n if and only if every permutation commuting with g is even.*

Proof. Let $Z(g, G)$ denote the elements of G commuting with g. Then the given condition is equivalent to

$$Z(g, S_n) = Z(g, A_n).$$

By Lagrange's Theorem

$$\#C = \frac{\#S_n}{\#Z(g, S_n)}$$

(considering the action $(g, x) \mapsto gxg^{-1}$ of G on $[g]$). Similarly if C' is the class of g in A_n then

$$\begin{aligned}
\#C' &= \frac{\#A_n}{\#Z(g, A_n)} \\
&= \frac{\#A_n}{\#Z(g, S_n)} \\
&= \frac{1}{2}\frac{\#S_n}{\#Z(g, S_n)} \\
&= \frac{\#C}{2}
\end{aligned}$$

It follows that C splits into two equal classes in A_n.

On the other hand, if there *is* an odd permutation in $Z(g, S_n)$ then it follows by the argument above that

$$\#C' > \frac{\#C}{2}.$$

But — again by the same argument — each A_n class in C contains at least $\#C/2$ elements.

It follows that C does not split in A_n. □

There are 4 even classes in S_5*:* $1^5, 2^2 1, 31^2, 5$*, containing* $1, 15, 20, 24$ *elements.*

The first 2 classes cannot split, since they contain an odd number of elements. Also 31^2 *does not split, since the odd permutation* (de) *commutes with the permutation* $(abc) \in 31^2$*.*

But since $24 \nmid 60$ *the last class must split into 2 classes* $5'$ *and* $5''$ *each containing 12 permutations.*

(b) *It follows that* A_5 *has 5 simple representations, of degrees* $1, a, b, c, d$, *say.*
Then

$$1^2 + a^2 + b^2 + c^2 + d^2 = 60,$$

ie

$$a^2 + b^2 + c^2 + d^2 = 59 \equiv 3 \bmod 8.$$

It follows that 3 of these 4 degrees, say a, b, c *are odd, while* $4 \mid d$. *(For* $n^2 \equiv 1 \bmod 8$ *if* n *is odd, while* $n^2 \equiv 4 \bmod 8$ *if* $n \equiv 2 \bmod 4$.*)*
Since $8^2 > 60$ *it follows that* $d = 4$, *and so*

$$a^2 + b^2 + c^2 = 43,$$

with $a, b, c \in \{1, 3, 5\}$.
We could show directly that the trivial representation is the only representation of degree 1. However, it is not necessary, since it is readily verified that the only solution is

$$a, b, c = 3, 3, 5.$$

Recall that the natural representation of S_n *splits into 2 simple parts* $1 + \sigma$. *Restricting* σ *to* A_n *gives the character*

	1^5	2^21	31^2	$5'$	$5''$
γ	4	0	1	-1	-1

Since

$$I(\gamma, \gamma) = \frac{1}{60}\left(1 \cdot 4^2 + 20 \cdot 1^2 + 12 \cdot (-1)^2 + 12 \cdot (-1)^2\right) = 1,$$

it follows that γ *is simple.*
At the moment the character table looks like

#	1	15	20	12	12
	1^5	2^21	31^2	$5'$	$5''$
1	1	1	1	1	1
α	3				
β	3				
γ	4	1	2	0	0
δ	5				

Note that if θ is a representation of G then $\det\theta$ is a 1-dimensional representation of G. (If θ takes the matrix form $g \mapsto T(g)$ then $g \mapsto \det T(g)$ under $\det\theta$.)

Since the trivial representation is the only 1-dimensional representation of A_5, it follows that $\det\theta = 1$ for $\theta = \alpha, \beta, \gamma$.

Consider the two 3-dimensional representations. If $g \in 2^2 1$ then $g^2 = 1$ and so g has eigenvalues ± 1 Since $\det\alpha = \det\beta = 1$, the eigenvalues are either $1, 1, 1$ or $1, -1, -1$.

If g has eigenvalues $1, 1, 1$ then $g \mapsto I$, and so $g \in \ker\alpha$. It follows that the subgroup generated by the class 31^2 lies in $\ker\alpha$. But this subgroup is normal, and so must be the whole of A_5 since A_5 is simple. (It is theorem that A_n is simple for all $n \geq 5$; but this is easy to establish for A_5, since a normal subgroup must be a union of classes, and no proper subset of $1, 15, 20, 12, 12$ including 1 has sum dividing 60, except $\{1\}$). Thus

$$\chi(2^2 1) = -1.$$

(Alternatively,

$$\sum |\chi(g)|^2 = \#G = 60,$$

since $I(\theta, \theta) = 1$ for a simple representation. If the eigenvalues were $1, 1, 1$ then $\chi(2^2 1) = 3$, and the 15 elements in this class would already contribute $15.3^2 = 135$ to the sum.)

Now suppose $g \in 31^2$. Then $g^3 = 1$, and so g has eigenvalues $\lambda, \mu, \nu \in \{1, \omega, \omega^2\}$, where $\omega = e^{2i\pi/3}$. Since $g \sim g^2$, if ω is an eigenvalue so is ω^2, and vice versa. Hence g has eigenvalues $1, 1, 1$ or $1, \omega, \omega^2$. The first is impossible, as above. Hence

$$\chi_\alpha(31^2) = \chi_\beta(31^2) = 1 + \omega + \omega^2 = 0.$$

Turning to the classes $5'$ and $5''$: suppose $g = (abcde) \in 5'$ has eigenvalues $\lambda, \mu, \nu \in \{1, \tau, \tau^2, \tau^3, \tau^4\}$, where $\tau = e^{2i\pi/5}$. Now $g \sim g^{-1}$ in A_5, since

$$(abcde) = x(edcba)x^{-1} \text{ with } x = (ae)(bd).$$

Thus if τ is an eigenvalue of g then so is τ^{-1}, and similarly if τ^2 is an eigenvalue of g then so is τ^{-2}.

The eigenvalues cannot be $1, 1, 1$, as above; so they are either $1, \tau, \tau^{-1}$ or $1, \tau^2, \tau^{-2}$.

Also $g \not\sim g^2$, *since otherwise* $\tau, \tau^2, \tau^3, \tau^4$ *would all be eigenvalues of* g. *It follows that* $g^2 \in 5''$.
Thus, on swapping the classes $5', 5''$ *if necessary, we have*

$$\chi_\alpha(5') = 1 + \tau + \tau^{-1},\ \chi_\alpha(5'') = 1 + \tau^2 + \tau^{-2}.$$

Since $\chi_\beta \neq \chi_\alpha$, *and all the other values of* χ_beta *are determined, we must have*

$$\chi_\beta(5') = 1 + \tau^2 + \tau^{-2},\ \chi_\beta(5'') = 1 + \tau + \tau^{-1}.$$

Note that if

$$\lambda = \tau + \tau^{-1}$$

then

$$\lambda^2 = \tau^2 + \tau^{-2} + 2.$$

Since

$$1 + \tau + \tau^2 + \tau^3 + \tau^4 = \frac{\tau^5 - 1}{\tau - 1} = 0,$$

it follows that

$$\tau^2 + \tau^{-2} = -1 - (\tau + \tau^{-1}).$$

Thus λ *satisfies the quadratic equation*

$$x^2 + x - 1 = 0,$$

and it is easy to see that the other root of this equation is $\mu = \tau^2 + \tau^{-2}$. *So*

$$\tau + \tau^{-1}, \tau^2 + \tau^{-2} = \frac{-1 \pm \sqrt{3}}{2}.$$

We have almost completed the character table:

#	1	15	20	12	12
	1^5	2^21	31^2	$5'$	$5''$
1	1	1	1	1	1
α	3	-1	0	$\frac{1+\sqrt{3}}{2}$	$\frac{1-\sqrt{3}}{2}$
β	3	-1	0	$\frac{1-\sqrt{3}}{2}$	$\frac{1+\sqrt{3}}{2}$
γ	4	0	1	-1	-1
δ	5				

It only remains to determine the 5-dimensional representation δ.

Recall that the regular representation ρ splits into simple parts

$$\rho = 1 + 3(\alpha + \beta) + 4\gamma + 5\delta,$$

while

$$\chi_\rho(g) = \begin{cases} \#G = 60 \text{ if } g = 1, \\ 0 \text{ if } g \neq 1, \end{cases} ,$$

This gives a simple way of completing a character table if all but one character is known.
Thus for any class $C \neq \{1\}$.

$$5\chi_\delta(C) = -1 - 3(\chi_\alpha(C) + \chi_\beta(C)) - 4\chi_\gamma(C),$$

So

$$\chi_\delta(2^2 1) = \frac{-1 - 2 \cdot 3 \cdot -1 - 4 \cdot 0}{4} = 1,$$
$$\chi_\delta(31^2) = \frac{-1 - 2 \cdot 3 \cdot 0 - 4 \cdot 1}{4} = -1,$$
$$\chi_\delta(5') = \frac{-1 - 3 \cdot -1 - 4 \cdot -1}{4} = 0,$$
$$\chi_\delta(5'') = \frac{-1 - 3 \cdot 1 - 4 \cdot -1}{4} = 0.$$

The table is complete:

#	1	15	20	12	12
	1^5	$2^2 1$	31^2	$5'$	$5''$
1	1	1	1	1	1
α	3	-1	0	$\frac{1+\sqrt{3}}{2}$	$\frac{1-\sqrt{3}}{2}$
β	3	-1	0	$\frac{1-\sqrt{3}}{2}$	$\frac{1+\sqrt{3}}{2}$
γ	4	0	1	-1	-1
δ	5	1	-1	0	0

Three remarks

(a) *The relation between the 2 3-dimensional representations α and β can be looked at in two different ways.*
First of all, if g is an odd permutation in S_5 then the map

$$\Theta : A_5 \to A_5 : x \mapsto gxg^{-1}$$

is an automorphim (a non-inner or outer automorphism) of A_5.

An automorphism Θ of a group G acts on the representations of G, sending each representation α into a representation $\alpha' = \Theta(\alpha)$ of the same dimension, given by

$$\alpha'(g) = \alpha(\Theta(g)).$$

If Θ is an inner automorphism then Θ sends each class into itself, and so $\Theta(\alpha) = \alpha$. So only outer automorphisms are of use in this context.

The outer automorphism of A_5 defined above swaps the classes $5'$ and $5''$, and so maps α into β, and vice versa.

(b) *Another way of looking at the relation between α and β is to apply galois theory. The cyclotomic extension $\mathbb{Q}(\tau)/\mathbb{Q}$ has galois group C_5, generated by the field automorphism $\tau \mapsto \tau^2$.*

This galois group acts on the representations of G, sending α into β and vice versa.

To be a little more precise (but going well outside the course), if χ is a character of a finite group G then $\chi(g) \in \bar{\mathbb{Q}}$, the field of algebraic numbers, since $\chi(g)$ is a sum of nth roots of unity, where $n = \#G$.

If $\mathbb{Q} \subset K \subset \bar{\mathbb{Q}}$ then any automorphism θ of K extends to an automorphism Θ of $\bar{\mathbb{Q}}$. It is not hard to see that any representation α of G can be expressed by matrices $A(g)$ with algebraic entries $A_{ij} \in \bar{\mathbb{Q}}$.

It follows that if the character table of G contains an irrational (but algebraic) entry like $\chi(C) = (1+\sqrt{3})/2$ then there will be another representation (of the same dimension) with entry $\chi'(C) = (1-\sqrt{3})/2$. So if G has only one representation χ of a given dimension then $\chi(C)$ must be rational for each class C.

Actually, we can go further: $\chi(C)$ is in fact an algebraic integer *— again because it is a sum of roots of unity. Now an algebraic integer that is rational is necessarily an ordinary integer. So if a rational number appears in a character table it must be an integer.*

This explains why the entries in character tables are mostly integers.

(c) *There are of course many ways of drawing up the character table of a finite group, one important tool being induced representations.*

In the case of the representations α, β of A_5, it is clear that we would have to start with some character of a subgroup involving

τ. An obvious choice is the 1-dimensional representation θ of $\langle(abcde)\rangle = C_5$ given by

$$(abcde) \mapsto \tau = e^{2\pi i/5}.$$

Inducing this up will give a representation $\Theta = \theta^{A_5}$ of A_5, of dimension $60/5 = 12$.
Recall the formula for this character:

$$\chi_\Theta([g]) = \frac{\#G}{\#H} \sum_{[h] \subset [g]} \frac{\#[h]}{\#[g]} \chi_\theta([h]).$$

Setting $g = (abcde)$,

$$2^2 1 \cap C_5 = \emptyset,\ 31^2 \cap C_5 = \emptyset,\ 5' \cap C_5 = \{g, g^{-1}),\ 5'' \cap C_5 = \{g^2, g^{-2}).$$

Hence

$$\chi_\Theta(5') = 12 \cdot \frac{1}{12}(\tau + \tau^{-1}) = \frac{-1 + \sqrt{3}}{2},$$

and similarly

$$\chi_\Theta(5'') = \frac{-1 - \sqrt{3}}{2},$$

Thus we have the character

	1^5	$2^2 1$	31^2	$5'$	$5''$
Θ	12	0	0	$\frac{-1+\sqrt{3}}{2}$	$\frac{-1-\sqrt{3}}{2}$

It is easy to see that

$$\Theta = \alpha + \gamma + \delta.$$

6. If α is a representation of the finite group G and β is a representation of the finite group H, define the representation $\alpha \times \beta$ of the product-group $G \times H$.

   Show that if α and β are simple then so is $\alpha \times \beta$, and show that every simple representation of $G \times H$ is of this form.

   Show that the symmetry group G of a cube is isomorphic to $C_2 \times S_4$.

   Into how many simple parts does the permutation representation of G defined by its action on the vertices of the cube divide?

   Answer:

(a) *If α, β are representations of G, H in the vector spaces U, V over k, then $\alpha \times \beta$ is the representation of $G \times H$ in the tensor product $U \otimes V$ defined by*

$$(g, h) \sum u \otimes v = \sum gu \otimes hv.$$

(b) *Evidently*

$$(g, h) \sim (g', h') \iff g \sim g', \ h \sim h'.$$

It follows that the conjugacy classes in $G \times H$ are

$$C \times D,$$

where C, D are classes in G, H. In particular if there are s classes in G and t classes in H then there are st classes in $G \times H$,

It follows that if $k = \mathbb{C}$ then $G \times H$ has st simple representations.

Evidently too

$$\chi_{\alpha\times\beta}(g, h) = \chi_\alpha(g)\chi_\beta(h).$$

Recall that (always assuming $k = \mathbb{C}$) a representation α is simple if and only if

$$I(\alpha, \alpha) = 1.$$

Now suppose α, β are simple. Then

$$\begin{aligned} I(\alpha \times \beta, \alpha \times \beta) &= \frac{1, \#G \times H}{\sum} {}_{(g,h)\in G\times H} \chi_{\alpha\times\beta}(g, h) \\ &= \frac{1, \#G\#H}{\sum} {}_{g\in G} \chi_\alpha(g) \sum_{h\in H} \chi_\beta(h) \\ &= 1 \times 1 = 1. \end{aligned}$$

Hence $\alpha \times \beta$ is simple.

Thus we have st simple representations of $G \times H$. They are different, since it follows by the same argument that

$$I(\alpha \times \beta, \alpha' \times \beta') = I(\alpha, \alpha')I(\beta, \beta') = 0$$

unless $\alpha = \alpha', \ \beta = \beta'$).

It follows that every simple representation of $G \times H$ is of the form $\alpha \times \beta$, with α, β simple.

(c) *It is easy to see that the symmetry group G of the cube has centre*

$$ZG = \{I, J\},$$

where J is reflection in the centre O of the cube.
Since J is improper it follows that

$$G = PG \times ZG,$$

where PG is the subgroup of proper symmetries of the cube.
Consider the action of PG on the 4 diagonals of the cube. This gives a homomorphism

$$\theta : PG \to S_4.$$

It is easy to see that the only symmetries sending each diagonal into itself (ie sending each vertex into itself or the antipodal vertex) are I, J. It follows that θ is injective.
The cube has 48 symmetries. For the subgroup S sending a given face into itself is isomorphic to D_4; and the symmetries sending this face into the other faces correspond to the left cosets of S. Hence S has order 8 and index 6, and so

$$\#G = 6 \cdot 8 = 48.$$

Just half of these cosets are improper (since Jg is improper if g is proper). Thus

$$\#PG = 24 = \#S_4.$$

Hence θ is bijective, and so

$$G = C_2 \times S_4.$$

(d) *Let γ be the representation of G defined by its action on the 8 vertices. Then*

$$\deg \gamma = 8.$$

Recall that S_4 has 5 classes $C = 1^4, 21^2, 2^2, 31, 4$, of sizes $1, 6, 3, 8, 6$, corresponding to these 5 cyclic types. Thus G has 10 classes $\{I\} \times C$, $\{J\} \times C$.
We know that a proper isometry in 3 dimensions leaving a point O fixed is a rotation about an axis through O.
It is easy to identify the 5 proper classes corresponding to $PG = S_4$ geometrically. Thus

- 1^4 *corresponds to* I*;*
- 21^2 *corresponds to the 6 half-turns about the axes joining mid-points of opposite edges;*
- 2^2 *corresponds to the 3 half-turns about the axes joining mid-points of opposite faces;*
- 31 *corresponds to the rotations through* $\pm 2\pi/3$ *about the 4 diagonals;*
- 4 *corresponds to the rotations through* $\pm\pi/2$ *about the axes joining mid-points of opposite faces.*

Each of the 5 improper classes is of the form JC*, where* C *is one of the proper classes, ie the rotations in* C *followed by reflection in the centre.*

Recall that if ρ *is the permutation representation arising from the action of a finite group* G *on a set* X *then*

$$\chi(g) = m,$$

the number of elements left fixed by g*.*

It is easy to determine the character of γ *from this:*

Class	$I \times 1^4$	$I \times 21^2$	$I \times 2^2$	$I \times 31$	$I \times 4$	$J \times 1^4$	$J \times 21^2$	$J \times 2^2$	$J \times 31$	$J \times 4$
Size	1	6	3	8	6	1	6	3	8	6
γ	8	0	0	2	0	0	4	0	0	0

Suppose

$$\gamma = n_1\sigma_1 + n_2\sigma_2 + \cdots + n_r\sigma_r.$$

Then

$$I(\gamma, \gamma) = n_1^2 + n_2^2 + \cdots + n_r^2.$$

From the table above

$$I(\gamma, \gamma) = \frac{1}{48}\left(1 \cdot 8^2 + 8 \cdot 2^2 + 6 \cdot 4^2\right) = 4.$$

Thus either

$$\gamma = 2\sigma \text{ or } \gamma = \sigma_1 + \sigma_2 + \sigma_3 + \sigma_4.$$

But from the table

$$I(1, \gamma) = \frac{1}{58}\left(1 \cdot 8 + 8 \cdot 2 + 6 \cdot 4\right) = 1.$$

(This can also be seen directly from the fact that if ρ is the representation arising from an action of G on X then $I(1, \rho)$ is equal to the number of orbits, in this case 1.) It follows that γ must split into 4 simple parts.

7. Show that every representation of a compact group is semisimple.

   Determine the simple representations of $\mathrm{U}(1)$.

   Verify that the simple characters of $\mathrm{U}(1)$ are orthogonal.

   Answer:

   (a) *We assume Haar's Theorem, that there exists an invariant measure dg on any compact group G; and we assume that this measure is strictly positive, ie*

   $$f(g) \geq 0 \forall g \implies \int f(g) dg \geq 0,$$

   with equality only if $f(g) = 0$ for all g.

   Now suppose α is a representation of G in V. Choose a positive-definite hermitian form $P(u, v)$ on V. Define the hermitian form $Q(u, v)$ by

   $$Q(u, v) = \int_G P(gu, gv)\, dg,$$

   where dg denotes the normalised Haar measure on G. Then Q is positive-definite and invariant under G.

   It follows that if $U \subset V$ is a stable subspace, then its orthogonal complement $U^\perp$ with respect to Q is also stable. Thus every stable subspace has a stable complement, and so the representation is semisimple.

   (b) *Since $\mathrm{U}(1)$ is abelian every simple representation α (over $\mathbb{C}$) is of degree 1; and since the group is compact*

   $$\operatorname{im} \alpha \subset \mathrm{U}(1),$$

   ie α is a homomorphism

   $$\mathrm{U}(1) \to \mathrm{U}(1).$$

   For each $n \in \mathbb{Z}$ the map

   $$E(n) : z \to z^n$$

defines such a homomorphism. We claim that every representation of U(1) *is of this form.*

For suppose

$$\alpha : U(1) \to U(1)$$

is a representation of U(1) *distinct from all the* $E(n)$*.*

Then

$$I(E_n, \alpha) = 0$$

for all n*, ie*

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \alpha(e^{i\theta}) e^{-in\theta}\, d\theta = 0.$$

In other words, all the Fourier coefficients of $\alpha(e^{i\theta})$ vanish.

But this implies (from Fourier theory) that the function itself must vanish, which is impossible since $\alpha(1) = 1$*.*

(c) *The invariant measure on* U(1) *(identified with the complex numbers* $e^{i\theta}$ *of absolute value 1) is*

$$\frac{1}{2\pi} d\,\theta.$$

Suppose $m \neq n$*. Then*

$$\begin{aligned} I(E(m), E(n)) &= \frac{1}{2\pi} \int_0^{2\pi} \overline{\chi_m(\theta)} \chi_n(\theta) d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{-im\theta} e^{in\theta} d\,\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)\theta} d\,\theta \\ &= \frac{1}{2\pi} \left[\frac{1}{i(n-m)} e^{i(n-m)\theta} \right]_0^{2\pi} \\ &= 0, \end{aligned}$$

ie $E(m), E(n)$ *are orthogonal.*

8. Show that SU(2) has just one simple representation of each degree $0, 1, 2, \ldots$.

   Determine the simple representations of U(2).

   Answer:

(a) *Suppose* $m \in \mathbb{N}$, *Let* $V(m)$ *denote the space of homogeneous polynomials* $P(z,w)$ *in* z, w. *Thus* $V(m)$ *is a vector space over* $\mathbb{C}$ *of dimension* $m+1$, *with basis* $z^m, z^{m-1}w, \ldots, w^m$.

Suppose $U \in \mathrm{SU}(2)$. *Then* U *acts on* z, w *by*

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathrm{SU}(2)$ *on* $V(m)$*:*

$$P(z,w) \mapsto P(z', w').$$

We claim that the corresponding representation of $\mathrm{SU}(2)$ *— which we denote by* $D_{m/2}$ *— is simple, and that these are the only simple (finite-dimensional) representations of* $\mathrm{SU}(2)$ *over* $\mathbb{C}$.

To prove this, let

$$\mathrm{U}(1) \subset \mathrm{SU}(2)$$

be the subgroup formed by the diagonal matrices

$$U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

The action of $\mathrm{SU}(2)$ *on* z, w *restricts to the action*

$$(z,w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathrm{U}(1)$. *Thus in the action of* $\mathrm{U}(1)$ *on* $V(m)$,

$$z^{m-r} w^r \mapsto e^{(m-2r)i\theta} z^{m-r} w^r,$$

It follows that the restriction of $D_{m/1}$ *to* $U(1)$ *is the representation*

$$D_{m/2} | \mathrm{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ *is the representation*

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathrm{U}(1)$.

Any $U \in \mathrm{SU}(2)$ *then* U *has eigenvalues* $e^{\pm i\theta}$ *(where* $\theta \in \mathbb{R}$*); and it is not difficult to show that*

$$U \sim U(\theta)$$

in SU(2). *It follows that the character of any representation of* SU(2)*, and therefore the representation itself, is completely determined by its restriction to the subgroup* U(1).
In particular, the character of $D_{m/2}$ is given by

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2}i\theta + \cdots + e^{-mi\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.
Now suppose $D_{m/2}$ is not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ is semisimple, since SU(2) *is compact.) Let a corresponding split of the representation space be*

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2}|\,\mathrm{U}(1)$ are distinct, the expression of $V(m)$ as a direct sum of U(1)*-spaces,*

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1}w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique. It follows that W_1 must be the direct sum of some of these spaces, and W_2 the direct sum of the others. In particular $z^m \in W_1$ or $z^n \in W_2$, say $z^m \in W_1$. Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z + w \\ -z + w \end{pmatrix}$$

under U. Hence

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r}w^r \rangle$, it follows that

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ of SU(2) *in $V(m)$ is simple.*
To see that every simple (finite-dimensional) representation of SU(2) *is of this form, suppose α is such a representation. Consider its restriction to* U(1). *Suppose*

$$\alpha|\,\mathrm{U}(1) = e_r E(r) + e_{r-1}E(r-1) + \cdots + e_{-r}E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α has character

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1} e^{(r-1)i\theta} + \cdots + e_{-r} e^{-ri\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.
Since $U(-\theta) \sim U(\theta)$ it follows that

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots .$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ with integer (possibly negative) coefficients:

$$\chi(\theta) = a_0\chi_0(\theta) + a_{1/2}\chi_{1/2}(\theta) + \cdots + a_s\chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

(b) *Each $U \in U(2)$ can be written as*

$$U = e^{i\theta} V$$

with $V \in \mathrm{SU}(2)$, since $|\det U| = 1$.
This gives a surjective homomorphism

$$\theta : \mathrm{U}(1) \times \mathrm{SU}(2) \to \mathrm{U}(2),$$

where we have identified $\mathrm{U}(1)$ with $\{z \in \mathbb{C} : |z| = 1\}$.
We have

$$\ker \theta = \{(1, I), (-1, -I)\},$$

since $U \in \mathrm{U}(1)$ can be written in two ways, as

$$U = e^{i\theta} V \text{ and } U = -e^{i\theta}(-V) = e^{i(\pi+\theta)}(-V).$$

It follows that the simple representations of $\mathrm{U}(2)$ arise from the simple representations α of $\mathrm{U}(1) \times \mathrm{SU}(2)$ which map $(-1, -I)$ to the identity.
Thus the simple representations of $\mathrm{U}(2)$ are

$$E(n) \times D(j),$$

where

$$n + 2j \equiv 0 \bmod 2.$$

9. Show that $\mathrm{SU}(2)$ is isomorphic to $\mathrm{Sp}(1)$ (the group of unit quaternions).

   Define a 2-fold covering $\mathrm{Sp}(1) \to \mathrm{SO}(3)$, and so determine the simple representations of $\mathrm{SO}(3)$.

   Answer:

   (a) *We can write each $q \in \mathbb{H}$ in the form*

   $$q = z + jw \quad (z, w \in \mathbb{C}),$$

   allowing us to identify $\mathbb{H}$ with $\mathbb{C}^2$. Note that

   $$jw = \bar{w} j$$

   for any $w \in \mathbb{C}$.
   Then $Q \in \mathrm{Sp}(1)$ acts on $\mathbb{H} = \mathbb{C}^2$ by left multiplication:

   $$\mu : q \mapsto Qq.$$

Suppose

$$Q = Z + jW.$$

Then

$$\begin{aligned} Q(z + jw) &= (Z + jW)(z + jw) \\ &= Zz + Zjw + jWz + jWjw \\ &= (Zz - \bar{W}w) + j(Wz + \bar{Z}w). \end{aligned}$$

In other words,

$$\begin{pmatrix} z & w \end{pmatrix} \mapsto \begin{pmatrix} Z & -\bar{W}//W & \bar{Z} \end{pmatrix} \begin{pmatrix} z & w \end{pmatrix},$$

ie

$$\mu(Q) = \begin{pmatrix} Z & -\bar{W}//W & \bar{Z} \end{pmatrix}$$

Also

$$|Q| = 1 \iff Z\bar{Z} + W\bar{W} = 1 \iff \mu(Q) \in \mathrm{SU}(2).$$

Thus we have established an isomorphism between $\mathrm{Sp}(1)$ *and* $\mathrm{SU}(2)$*.*

(b) The quaternion

$$q = t + xi + yj + zk \quad (t, x, y, z \in \mathbb{R})$$

is said to be purely imaginary if $t = 1$*. This is the case if and only if*

$$\bar{q} = -q.$$

The purely imaginary quaternions form a 3-dimensional vector space V *over* $\mathbb{R}$*.*
If $Q \in \mathbb{H},\ v \in V$ *then*

$$(QvQ^a st)^* = Qv^*Q^* = -QvQ^*.$$

Thus

$$QvQ^* \in V.$$

Thus each $Q \in \mathbb{H}$ *defines a linear map*

$$\theta(Q) : V \to V$$

under which

$$v \mapsto QvQ^*.$$

It is a straightforward matter to verify that

$$\theta(Q_1Q_2) = \theta(Q_1)\theta(Q_2),$$

so that if $Q \in \mathrm{Sp}(1)$ *then*

$$\theta(Q)\theta(Q^*) = I,$$

establishing that the map under which $Q \in \mathrm{Sp}(1)$ *acts on* V *by*

$$v \mapsto QvQ^*$$

is a homomorphism

$$\Theta : \mathrm{Sp}(1) \to \mathrm{GL}(V) = \mathrm{GL}(\mathbb{R}, 3).$$

Suppose $v \in V,\ |v| = 1$*. Then* $v \in \mathrm{Sp}(1)$*; and if* $T = \Theta(v)$ *then*

$$Tv = v^2v^* = v.$$

Thus $\Theta(v)$ *is a rotation about the axis* v*; and since* $v^2 = -|v| = -1$*,* v *is in fact a half-turn about this axis.*
Since half-turns generate $\mathrm{SO}(3)$ *it follows that* Θ *is surjective.*
Suppose $Q \in \ker\Theta$*, ie*

$$QvQ^* = v$$

for all $v \in V$*, ie*

$$Qv = vQ$$

for all v*. Since* $Qt = tQ$ *for all* $t \in \mathbb{R}$*, it follows that*

$$Qq = qQ$$

for all $qin\mathbb{H}$*, ie*

$$Q \in ZH = \{\pm 1\}.$$

Thus

$$\ker\Theta = \{\pm 1\}.$$

Thus the simple representations of $\mathrm{SO}(3)$ *correspond to the representations* $D(j)$ *of* $\mathrm{SU}(2)$ *which act trivially on* $-I$*. It is easy to see that these are the* $D(j)$ *with* $j \in \mathbb{N}$