Course 3413 — Group Representations
Sample Paper I

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Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are compact (or finite), and all representations are of finite degree over $\mathbb{C}$.

1. What is a *group representation*?

   What is meant by saying that a representation is *simple*?

   Determine from first principles all simple representations of $S(3)$.

   Answer:

   (a) A representation α of a group G in a vector space V is a homomorphism
   $$\alpha : G \to \mathrm{GL}(V).$$

   (b) The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if
   $$g \in G, u \in U \implies gu \in U.)$$

(c) We have

$$S_3 = \langle s, t : s^3 = t^2 = 1, st = ts^2 \rangle$$

(taking $s = (abc), t = (ab)$).

Let us first suppose α is a 1-dimensional representations of S_3. ie a homomorphism

$$\alpha : S_3 \to \mathbb{C}^*.$$

Suppose

$$\alpha(s) = \lambda,\ \alpha(t) = \mu.$$

Then

$$\lambda^3 = \mu^2 = 1,\ \lambda\mu = \mu\lambda^2.$$

The last relation gives

$$\lambda = 1.$$

Thus there are just two 1-dimensional representations given by

$$s \mapsto 1, t \mapsto \pm 1.$$

Now suppose α is a simple representation of S_3 in the vector space V over $\mathbb{C}$, where $\dim V \geq 2$. Let $e \in V$ be an eigenvector of s:

$$se = \lambda e;$$

and let

$$f = te.$$

Then

$$sf = ste = ts^2e = \lambda^2 te = \lambda^2 f,$$

ie f is a λ^2-eigenvector of s.

It follows that the subspace

$$\langle e, f \rangle \subset V$$

is stable under S_3, since

$$se = \lambda e,\ sf = \lambda^2 f,\ te = f,\ tf = t^2 e = e.$$

Since V by hypothesis is simple, it follows that

$$V = \langle e, f \rangle.$$

In particular, $\dim \alpha = 2$, and e, f form a basis for V.

Since $s^3 = 1$ we have $\lambda^3 = 1$, ie $\lambda \in \{1, \omega, \omega^2\}$, where $\omega = e^{2\pi i/3}$.

If $\lambda = 1$ then s would have eigenvalues $1, 1$ (since $1^3 = 1$). But we know that s (ie $\alpha(s)$) is diagonalisable. It follows that $s = I$.

Thus s will be diagonal with respect to any basis. Since we can always diagonalise t, we can diagonalise s, t simultaneously. But in that case the representation would not be simple; for if e is a common eigenvector of s, t then the 1-dimensional space $\langle e \rangle$ is stable under S_3.

Thus we are left with the cases $\lambda \in \{\omega, \omega^2\}$. If $\lambda = \omega^2$ then on swapping e and f we would have $\lambda = \omega$. So we have only one 2-dimensional representation (up to equivalence):

$$s \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \; t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In conclusion, S_3 has just 3 simple representations: two of dimension 1, and one of dimension 2.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Show that the natural representation of S_n in $\mathbb{C}^n$ (by permutation of coordinates) splits into 2 simple parts, for any $n > 1$.

   Answer:

   (a) *The representation α of G in V is said to be semisimple if it can be expressed as a sum of simple representations:*

   $$\alpha = \sigma_1 + \cdots + \sigma_m.$$

   This is equivalent to the condition that each stable subspace $U \subset V$ has a stable complement W:

   $$V = U \oplus W.$$

   (b) *Suppose α is a representation of the finite group G in the vector space V over $\mathbb{C}$. Let*

   $$P(u, v)$$

   be a positive-definite hermitian form on V. Define the hermitian form Q on V by

   $$Q(u, v) = \frac{1}{\|G\|} \sum_{g \in G} H(gu, gv).$$

Then Q is positive-definite (as a sum of positive-definite forms). Moreover Q is invariant under G, ie

$$Q(gu, gv) = Q(u, v)$$

for all $g \in G, u, v \in V$. For

$$\begin{aligned} Q(hu, hv) &= \frac{1}{\|G\|} \sum_{g \in G} H(ghu, ghv) \\ &= \frac{1}{|G|} \sum_{g \in G} H(gu, gv) \\ &= Q(u, v), \end{aligned}$$

since gh runs over G as g does.
Now suppose U is a stable subspace of V. Then

$$U^{\perp} = \{v \in V : Q(u, v) = 0 \forall u \in U\}$$

is a stable complement to U.

(c) *It is evident that the subspaces*

$$U = \{(x, x, \ldots, x) : x \in \mathbb{C}\},\ V = \{(x_1, x_2, \ldots, x_n) : x_1 + x_2 + \cdots + x_n\}$$

of $\mathbb{C}^n$ are stable under S_n. We shall show that they are simple, and that

$$\mathbb{C}^n = U \oplus V,$$

from which the result follows.
U is simple, since $\dim U = 1$.
Suppose $v = (x_1, \ldots, x_n) \in V$ is non-zero. The x_i cannot all be equal, since their sum is zero. Suppose $x_i \neq x_j$, where $i < j$. Then

$$v - \pi_{ij} v = (x_i - x_j) e_{ij} \in V,$$

where

$$e_{ij} = (0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0),$$

with 1 in position i and -1 in position j.
There is a permutation σ taking the pair i, j into any other pair i', j'. It follows that

$$e_{ij} \in V$$

for all i, j.

But the e_{ij} *span* V; *for if* $v = (x_1, \ldots, x_n) \in V$ *then*

$$v = x_1 e_{1n} + x_2 e_{2n} + \cdots + x_{n-1} e_{n-1,n}.$$

Thus V *is simple.*

Finally,

$$U \cap V = 0,$$

since

$$v = (x, x, \ldots, x) \in V \implies x = 0,$$

while

$$U + V = \mathbb{C}^n,$$

since

$$(x_1, \ldots, x_n) = (s, \ldots, s) + (x_1 - s, \ldots, x_n - s),$$

where

$$s = (x_1 + \cdots + x_n)/n.$$

Thus $\mathbb{C}^n$ *is a direct sum of simple subspaces, and so is semisimple.*

3. Determine the conjugacy classes in S_4, and draw up its character table.

   Determine also the representation-ring for S_4, ie express the product $\alpha\beta$ of each pair of simple representations as a sum of simple representations.

   Answer:

   (a) *S_4 has 5 classes, corresponding to the types* $1^4, 1^2 2, 13, 2^2, 4$. *Thus S_4 has 5 simple representations.*

   Each symmetric group S_n (for $n \geq 2$) has just 2 1-dimensional representations, the trivial representation 1 *and the parity representation* ϵ.

   Let $S_4 = \mathrm{Perm}(()X)$, *where* $X = \{a, b, c, d\}$. *The action of S_4 on X defines a 4-dimensional representation ρ of S_4, with character*

   $$\chi(g) = |\{x \in X : gx = x\}|$$

   In other words $\chi(g)$ is just the number of 1-cycles in g.

So now we can start our character table (where the second line gives the number of elements in the class):

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
ρ	4	2	1	0	0

Now

$$I(\rho,\rho) = \frac{1}{24}(1\cdot 16 + 6\cdot 4 + 8\cdot 1) = 2.$$

It follows that ρ has just 2 simple parts. Since

$$I(1,\rho) = \frac{1}{24}(1\cdot 4 + 6\cdot 2 + 8\cdot 1) = 1,$$

It follows that

$$\rho = 1 + \alpha,$$

where α is a simple 3-dimensional representation, with character given by

$$\chi(g) = \chi_\rho(g) - 1.$$

The representation $\epsilon\alpha$ is also simple, and is not equal to α since it has a different character. So now we have 4 simple characters of S_4, as follows:

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1

To find the 5th simple representation, we can consider α^2. This has character

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
α^2	9	1	0	1	1

We have

$$\begin{aligned}
I(1,\alpha^2) &= \frac{1}{24}(9+6+3+6) = 1,\\
I(\epsilon,\alpha^2) &= \frac{1}{24}(9-6+3-6) = 0,\\
I(\alpha,\alpha^2) &= \frac{1}{24}(27+6-3-6) = 1,\\
I(\epsilon\alpha,\alpha^2) &= \frac{1}{24}(27-6-3+6) = 1. I(\alpha^2,\alpha^2) \quad = \frac{1}{24}(81+6+3+6) = 4,
\end{aligned}$$

It follows that α^2 has 4 simple parts, so that

$$\alpha^2 = 1 + \alpha + \epsilon\alpha + \beta,$$

where β is the 5th simple representation, with character given by

$$\chi_\beta(g) = \chi_\alpha(g)^2 - 1 - \chi_\alpha(g) - \epsilon(g)\chi_\alpha(g).$$

This allows us to complete the character table:

	1^4 (1)	$1^2 2$ (6)	13 (8)	2^2 (3)	4 (6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1
β	2	0	-1	2	0

(b) *We already know how to express α^2 in terms of the 5 simple representations. Evidently $\epsilon\beta = \beta$ since there is only 1 simple representation of dimension 2. The character of $\alpha\beta$ is given by*

	1^4	$1^2 2$	13	2^2	4
$\alpha\beta$	6	0	0	-2	0

We have

$$I(\alpha\beta,\alpha\beta) = \frac{1}{24}(36+12) = 2.$$

Thus $\alpha\beta$ has just 2 simple parts. These must be α and $\epsilon\alpha$ to give dimension 6:

$$\alpha\beta = \alpha + \epsilon\alpha.$$

Also we have

$$I(\beta^2, \beta^2) = \frac{1}{24}(16 + 8 + 48) = 3.$$

Thus β has 3 simple parts. So by dimension, we must have

$$\beta^2 = 1 + \epsilon + \beta.$$

Now we can give the multiplication table for the representation-ring:

	1	ϵ	β	α	$\epsilon\alpha$
1	1	ϵ	β	α	$\epsilon\alpha$
ϵ	ϵ	1	β	$\epsilon\alpha$	α
β	β	β	$1+\epsilon+\beta$	$\alpha+\epsilon\alpha$	$\alpha+\epsilon\alpha$
α	α	$\epsilon\alpha$	$\alpha+\epsilon\alpha$	$1+\beta+\alpha+\epsilon\alpha$	$\epsilon+\beta+\alpha+\epsilon\alpha$
$\epsilon\alpha$	$\epsilon\alpha$	α	$\alpha+\epsilon\alpha$	$\epsilon+\beta+\alpha+\epsilon\alpha$	$1+\beta+\alpha+\epsilon\alpha$

4. Prove that the number of simple representations of a finite group G is equal to the number of conjugacy classes in G.

Answer: *Let the simple representations of G be $\sigma_1, \ldots, \sigma_r$; and let $\chi_i(g)$ be the character of σ_i.*

The simple characters $\chi_1, \ldots, \chi_r$ are linearly independent. For if say

$$\rho_1\chi_1(g) + \cdots + \rho_s\chi_s(g) = 0$$

it follows from the formula for the intertwining number that for any representation α

$$\rho_1 I(\alpha, \sigma_1) + \cdots + \rho_r I(\alpha, \sigma_r) = 0.$$

But on applying this with $\alpha = \sigma_i$ we deduce that $\rho_i = 0$ for each i.

The characters are class functions:

$$\chi(gxg^{-1}) = \chi(x).$$

The space of class functions has dimension s, the number of classes in G. It follows that $r \le s$.

To prove that $r = s$, it is sufficient to show that the characters span the space of class functions.

Suppose $g \in G$ has order e. Let $[g]$ denote the class of g, and let $C = \langle g \rangle$ be the cyclic group generated by g.

The group C has e 1-dimensional representations $\theta_1, \ldots, \theta_e$ given by

$$\theta_i : g \mapsto \omega^i,$$

where $\omega = e^{2\pi i/e}$.

Let

$$f(x) = \theta_0(x) + \omega^{-1}\theta_1(x) + \omega^{-2}\theta_2(x) + \cdots + \omega^{-e+1}\theta_{e-1}(x)$$

Then

$$f(g^j) = \begin{cases} e & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Now let us "induce up" each of the characters θ_i from C to G. We have

$$\theta_i^G(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} \theta_i(y).$$

Let $F(x)$ be the same linear combination of the induced characters that $f(x)$ was of the θ_i. Then

$$F(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} f(y).$$

Since $f(y)$ vanishes away from g, we deduce that $F(x)$ vanishes off the class $[g]$, and is non-zero on that class:

$$F(x) \begin{cases} > 0 & \text{if } x \in [g], \\ = 0 & \text{if } x \notin [g]. \end{cases}$$

It follows that every class function on G can be expressed as a linear combination of characters, and therefore as a linear combination of simple characters. Hence the number of simple characters is at least as great as the number of classes.

We have shown therefore that the number of simple representations is equal to the number of classes.

5. Show that if the finite group G has simple representations $\sigma_1, \ldots, \sigma_s$ then

$$\deg^2 \sigma_1 + \cdots + \deg^2 \sigma_s = |G|.$$

Determine the degrees of the simple representations of S_6.

Answer:

(a) *Consider the regular representation ρ of G. We have*

$$\chi_\rho(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Thus if α is any representation of G,

$$I(\rho, \alpha) = \chi_\alpha(e) = \dim \alpha.$$

Applying this to the simple representations $\alpha = \sigma_i$ we deduce that

$$\rho = (\dim \sigma_1)\sigma_1 + \cdots + (\dim \sigma_s)\sigma_s.$$

Taking dimensions on each side,

$$|G| = (\dim \sigma_1)^2 + \cdots + (\dim \sigma_s)^2.$$

(b) *S_6 has 11 classes:*

$$1^6,\ 21^4,\ 2^21^2,\ 2^3,\ 31^3,\ 321,\ 3^2,\ 3^2,\ 41^2,\ 51,\ 6.$$

Hence it has 11 simple representations over $\mathbb{C}$.
It has 2 representations of degree 1: 1 and the parity representation ϵ.
The natural representation ρ_1 of degree 6 (by permutation of coordinates) splits into two simple parts:

$$\rho_1 = 1 + \sigma_1,$$

where σ_1 is of degree 5.
If α is a simple representation of odd degree, then

$$\epsilon\alpha \neq \alpha.$$

For a transposition t has eigenvalues ± 1, since $t^2 = 1$. Hence

$$\chi_\alpha(t) \neq 0.$$

But

$$\chi_{\epsilon\alpha}(t) = \chi_\epsilon(t)\chi_\alpha(t) = -\chi_\alpha(t).$$

Thus the simple representations of odd degree d divide into pairs α, $\epsilon\alpha$. So there are an even number of representations of degree d.

In particular there are at least 2 simple representations of degree 5: σ and $\epsilon\sigma$.

We are going to draw up a partial character table for S_6, adding rows as we gather more material.

	1^6	21^4	2^21^2	2^3	31^3	321	3^2	42	41^2	51	6
$\#$	1	15	45	15	40	120	40	90	90	144	120
ρ_1	6	4	2	0	3	1	0	2	0	1	0
σ_1	5	3	1	-1	2	0	-1	1	-1	0	-1
ρ_2	15	7	3	3	3	1	0	1	1	0	0
τ	14	6	2	2	2	0	-1	0	0	-1	-1
σ_2	9	3	1	3	0	0	0	-1	1	-1	0
ρ_3	20	8	4	0	2	2	2	0	0	0	0
θ	19	7	3	-1	1	1	1	-1	-1	-1	-1
σ_3	5	1	1	-3	-1	1	2	-1	-1	0	0
σ_1^2	25	9	1	1	4	0	1	1	1	0	1
ϕ	24	8	0	0	3	-1	0	0	0	-1	0

Now consider the permutation representation ρ_2 arising from the action of S_6 on the 15 pairs of elements. Evidently

$$I(\rho_2, 1) > 0,$$

since all the terms in the sum for this are ≥ 0. Let $\tau = \rho_2 - 1$. Then

$$I(\tau, \tau) = \frac{1}{720}(196 + 540 + 180 + 60 + 160 + 40 + 144 + 120) = 2,$$

while

$$I(\tau, \sigma_1) = \frac{1}{720}(70 + 270 + 90 - 30 + 160 + 40 + 120) = 1.$$

Thus

$$\sigma_2 = \tau - \sigma_1$$

is simple.

So far we have 6 simple representations:

$$1, \epsilon, \sigma_1, \epsilon\sigma_1, \sigma_2, \epsilon\sigma_2,$$

of degrees 1,1,5,5,9,9.

Next consider the permutation representation ρ_3 arising from the action of S_6 on the 20 subsets of 3 elements. Evidently

$$I(\rho_3, 1) > 0,$$

since all the terms in the sum for this are ≥ 0.
[Although not needed here, it is worth recalling that if ρ is a permutation representation arising from the action of G on the set X then $I(\rho, 1)$ is equal to the number of orbits of the action.]
Let $\theta = \rho_3 - 1$. Then

$$I(\theta, \theta) = \frac{1}{720}(361+735+405+15+40+120+40+90+90+144+120) = 3.$$

Thus θ has 3 simple parts.
Now

$$I(\theta, \sigma_1) = \frac{1}{720}(95+315+135+15+80-40-90+90+120) = 1,$$

while

$$I(\theta, \sigma_2) = \frac{1}{720}(171+315+135-45+90-90+144) = 1.$$

It follows that

$$\sigma_3 = \theta - \sigma_1 - \sigma_2$$

is simple.
Now we have 8 simple representations:

$$1, \epsilon, \sigma_1, \epsilon\sigma_1, \sigma_3, \epsilon\sigma_3, \sigma_2, \epsilon\sigma_2,$$

of degrees 1,1,5,5,5,5,9,9.
We have 3 remaining simple representations. Suppose they are of degrees a, b, c. Then

$$720 = 2 \cdot 1^2 + 4 \cdot 5^2 + 2 \cdot 9^2 + a^2 + b^2 + c^2$$

ie

$$a^2 + b^2 + c^2 = 456.$$

Now

$$456 \equiv 0 \bmod 8.$$

If n is odd then $n^2 \equiv 1 \bmod 8$. It follows that a, b, c are all even, say

$$a = 2d,\ b = 2e,\ c = 2f,$$

with

$$d^2 + e^2 + f^2 = 114.$$

Since

$$114 \equiv 2 \bmod 8,$$

it follows that two of d, e, f are odd and one is divisible by 4. Let us suppose these are d, e, f in that order. Then

$$f \in \{4, 8\}.$$

If $f = 4$ then

$$d^2 + e^2 = 98 \implies d = e = 7,$$

while if $f = 8$ then

$$d^2 + e^2 = 50 \implies d = e = 5.$$

So the three remaining simple representations have degrees

$$8, 14, 14 \text{ or } 10, 10, 16.$$

Let

$$\phi = \sigma_1^2 - 1.$$

Then

$$I(\phi, \phi) = \frac{1}{720}(576 + 960 + 360 + 120 + 144) = 3.$$

Also

$$I(\phi, \sigma_1) = \frac{1}{720}(120 + 360 + 240) = 1,$$

while

$$I(\phi, \sigma_2) = \frac{1}{720}(216 + 360 + 144) = 1.$$

Thus

$$\sigma_4 = \phi - \sigma_1 - \sigma_2$$

is a simple representation of degree 10.

We conclude that the 11 simple representations have degrees

$$1, 1, 5, 5, 5, 5, 9, 9, 10, 10, 16.$$

6. Show that the dodecahedron has 60 proper symmetries, and determine how these are divided into conjugacy classes.

Answer:

(a) *Let G be the symmetry group of the dodecahedron.*

The symmetries sending a given face $F = ABCDE$ into itself form a subgroup $D_5 \subset G$ of order 10.

Suppose two symmetries $g, h \in G$ send F into the same face. Then

$$gF = hF \iff h^{-1}gF = F \iff h^{-1}g \in S \iff hS = gS$$

Thus there is a 1-1 correspondence between the left cosets of S and the faces of the dodecahedron. Hence the index

$$[G : H] = 12,$$

the number of faces; and so G has order

$$12 \cdot 10 = 120.$$

(b) *The centre of G is*

$$ZG = \{I, J\},$$

where J is reflection in the centre of the cube.

Since J is improper, it follows that the proper symmetries form a subgroup P of order 60; and if C is a class of proper symmetries then JC is a class of improper symmetries.

A proper isometry in 3 dimensions leaving a point O fixed is necessarily a rotation about an axis through O. Hence any proper symmetry of the cube is a rotation about some axis through the centre O of the cube.

Let ℓ be the axis joining the centres of a pair F, F' of opposite faces. There are 4 rotations about ℓ (apart from the identity I) through angles $\pm 2\pi/5, \pm 4\pi/5$ (each of order 5) sending the dodecahedron into itselfs. Since there are 6 pairs of opposite faces, these give $6 \cdot 4 = 24$ symmetries.

It is easy to see that the rotations through angles $\pm 2\pi/5$ are all conjugate, giving a class of size 12, since if we reverse the direction of the axis, a rotation through θ becomes a rotation through $-\theta$.

Similarly the rotations through $\pm 4\pi/5$ are conjugate, giving another class of size 12.

There are 3 faces meeting each vertex, so there are rotations through angles $\pm 2\pi/3$ about each axis joining opposite vertices. These are all conjugate, giving a class of size $10 \cdot 2 = 20$, since there are 10 pairs of opposite vertices.

Finally, there are half-turns (rotations through π) about the axis joining the centres of opposite edges; and these are all conjugate.

Since there are 15 pairs of opposite edges, these form a class of size 15.

There is a fifth proper class $\{I\}$ of size 1.

Thus there are 5 proper classes, of sizes 1,12,12,15,20.

Similarly there are 5 improper classes, of the same sizes, since there is an improper class JC corresponding to each proper class C.

7. Define a *measure* μ on a compact space X.

   Sketch the proof that if G is a compact group then there exists a unique invariant measure on G with $\mu(1) = 1$.

   Answer:

   (a) *A* measure μ *on X is a continuous linear functional*

   $$\mu : C(X) \to \mathbb{C},$$

   where $C(X) = C(X, \mathbb{R})$ is the space of real-valued continuous functions on X with norm $\|f\| = \sup |f(x)|$.

   (b) *The compact group G acts on $C(G)$ by*

   $$(gf)(x) = f(g^{-1}x).$$

   The measure μ is said to be invariant under G if

   $$\mu(gf)\mu(f)$$

   for all $g \in G,\ f \in C(G)$.

   By an average *F of $f \in C(G)$ we mean a function of the form*

   $$F = \lambda_1 g_1 f + \lambda_2 g_2 f + \cdots + \lambda_r g_r f,$$

   where $0 \leq \lambda_i \leq 1,\ \sum \lambda_i = 1$ and $g_1, g_2, \ldots, g_r \in G$.

   If F is an average of f then

   i. $\inf f \leq \inf F \leq \sup F \leq supf$;

ii. If μ is an invariant measure then $\mu(F) = \mu(f)$;

iii. An average of F is an average of f.

If we set

$$\mathrm{var}(f) = \sup f - \inf f$$

then

$$\mathrm{var}(F) \leq \mathrm{var}(f)$$

for any average F of f. We shall establish a sequence of averages $F_0 = f, F_1, F_2, \ldots$ (each an average of its predecessor) such that $\mathrm{var}(F_i) \to 0$. *It follows that*

$$F_i \to c \in \mathbb{R},$$

ie $F_i(g) \to c$ for each $g \in G$.

Suppose $f \in C(G)$. It is not hard to find an average F of f with $\mathrm{var}(F) < \mathrm{var}(f)$. *Let*

$$V = \{g \in G : f(g) < \frac{1}{2}(\sup f + \inf f),$$

ie V is the set of points where f is 'below average'. Since G is compact, we can find $g_1, \ldots, g_r$ such that

$$G = g_1 V \cup \cdots \cup g_r V.$$

Consider the average

$$F = \frac{1}{r}\left(g_1 f + \cdots + g_r f\right).$$

Suppose $x \in G$. Then $x \in g_i V$ for some i, ie

$$g_i^{-1} x \in V.$$

Hence

$$(g_i f)(x) < \frac{1}{2}(\sup f + \inf f),$$

and so

$$\begin{aligned} F(x) &< \frac{r-1}{r} \sup f + \frac{1}{2r}(\sup f + \inf f) \\ &= \sup f - \frac{1}{2r} \sup f - \inf f. \end{aligned}$$

Hence $\sup F < \sup f$ *and so*

$$\mathrm{var}(F) < \mathrm{var}(f).$$

This allows us to construct a sequence of averages $F_0 = f, F_1, F_2, \ldots$ *such that*

$$\mathrm{var}(f) = \mathrm{var}(F)_0 > \mathrm{var}(F)_1 > \mathrm{var}(F)_2 > \cdots.$$

But that is not sufficient to show that $\mathrm{var}(F)_i \to 0$. *For that we must use the fact that any* $f \in C(G)$ *is* uniformly *continuous.*

[I would accept this last remark as sufficient in the exam, and would not insist on the detailed argument that follows.]

In other words, given $\epsilon > 0$ *we can find an open set* $U \ni e$ *such that*

$$x^{-1}y \in U \implies |f(x) - f(y)| < \epsilon.$$

Since

$$(g^{-1}x)^{-1}(g^{-1}y) = x^{-1}y,$$

the same result also holds for the function gf. *Hence the result holds for any average* F *of* f.

Let V *be an open neighbourhood of* e *such that*

$$VV \subset U, \quad V^{-1} = V.$$

(If V *satisfies the first condition, then* $V \cap V^{-1}$ *satisfies both conditions.) Then*

$$xV \cup yV \neq \emptyset \implies |f(x) - f(y)| < \epsilon.$$

For if $xv = yv'$ *then*

$$x^{-1}y = vv'^{-1} \in U.$$

Since G *is compact we can find* $g_1, \ldots, g_r$ *such that*

$$G = g_1V \cup \cdots \cup g_rV.$$

Suppose f *attains its minimum* $\inf f$ *at* $x_0 \in g_iV$*; and suppose* $x \in g_jV$. *Then*

$$g_i^{-1}x_0,\ g_j^{-1}x \in V.$$

Hence

$$\left(g_j^{-1}x\right)^{-1}\left(g_i^{-1}x_0\right) = \left(g_ig_j^{-1}x\right)^{-1}x_0 \in U,$$

and so
$$|f(g_i g_j^{-1} x) - f(x_0)| < \epsilon.$$
In particular,
$$(g_j g_i^{-1} f)(x) < \inf f + \epsilon.$$
Let F be the average
$$F = \frac{1}{r^2} \sum_{i,j} g_j g_i^{-1} f.$$
Then
$$\sup F < \frac{r^2 - 1}{r^2} \sup f + \frac{1}{r^2}(\inf f + \epsilon),$$
and so
$$\begin{aligned} \mathrm{var}(F) &< \frac{r^2 - 1}{r^2} \mathrm{var}(f) + \frac{1}{r^2}\epsilon \\ &< \frac{r^2 - 1/2}{r^2} \mathrm{var}(f), \end{aligned}$$
if $\epsilon < \mathrm{var}(f)/2$.
Moreover this result also holds for any average of f in place of f. It follows that a succession of averages of this kind
$$F_0 = f, F_1, \ldots, F_s$$
will bring us to
$$\mathrm{var}(F)_s < \frac{1}{2} \mathrm{var}(f).$$
Now repeating the same argument with F_s, and so on, we will obtain a sequence of successive averages $F_0 = f, F_1, \ldots$ with
$$\mathrm{var}(F)_i \downarrow 0.$$
It follows that
$$F_i \to c$$
(the constant function with value c).
It remains to show that this limit value c is unique. For this we introduce right averages
$$H(x) = \sum_j \mu_j f(x h_j)$$

where $0 \leq \mu_j \leq 1$, $\sum \mu_j = 1$. *(Note that a right average of* f *is in effect a left average of* $\tilde{f}$, *where* $\tilde{f}(x) = f(x^{-1})$. *In particular the results we have established for left averages will hold equally well for right averages.)*

Given a left average and a right average of f, *say*

$$F(x) = \sum \lambda_i f(g_i^{-1}x), \quad H(x) = \sum \mu_j f(xh_j),$$

we can form the joint average

$$J(x) = \sum_{i,j} \lambda_i \mu_j f(g_i^{-1}xh_j).$$

It is easy to see that

$$\begin{aligned} &\inf F \leq \inf J \leq \sup J \leq \sup H, \\ &\sup F \geq \sup J \geq \inf J \geq \inf H. \end{aligned}$$

But if now $H_0 = f, H_1, \ldots$ *is a succession of right averages with* $H_i \to d$ *then it follows that*

$$c = d.$$

In particular, any two convergent sequences of successive left averages must tend to the same limit. We can therefore set

$$\mu(f) = c.$$

Thus $\mu(f)$ *is well-defined; and it is invariant since* f *and* gf *have the same set of averages. Finally, if* $f = 1$ *then* $\operatorname{var}(f) = 0$, *and* $f, f, f, \ldots$ *converges to 1, so that*

$$\mu(1) = 1.$$

The invariant measure on G *is unique up to a scalar multiple. In other words, it is unique if we normalise the measure by specifying that*

$$\mu(1) = 1$$

(where 1 *on the left denotes the constant function 1).*

8. Determine the simple representations of $\mathrm{SO}(2)$.

   Determine the simple representations of $\mathrm{O}(2)$.

   Answer:

(a) *Let*

$$R(\theta) \in \mathrm{SO}(2)$$

denote rotation through angle θ*. Then the map*

$$R(\theta) \mapsto e^{i\theta} : \mathrm{SO}(2) \to \mathrm{U}(1)$$

is an isomorphism, allowing us to identify $\mathrm{SO}(2)$ *with* $\mathrm{U}(1)$.

This group is abelian; so every simple representation α *(over* $\mathbb{C}$*) is of degree 1; and since the group is compact*

$$\operatorname{im} \alpha \subset \mathrm{U}(1).$$

ie α *is a homomorphism*

$$\mathrm{U}(1) \to \mathrm{U}(1).$$

For each $n \in \mathbb{Z}$ *the map*

$$E(n) : z \to z^n$$

defines such a homomorphism. We claim that every representation of $\mathrm{U}(1)$ *is of this form.*

For suppose

$$\alpha : U(1) \to U(1)$$

is a representation of $\mathrm{U}(1)$ *distinct from all the* $E(n)$*.*

Then

$$I(E_n, \alpha) = 0$$

for all n*, ie*

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \alpha(e^{i\theta}) e^{-in\theta} \, d\theta = 0.$$

In other words, all the Fourier coefficients of $\alpha(e^{i\theta})$ vanish.

But this implies (from Fourier theory) that the function itself must vanish, which is impossible since $\alpha(1) = 1$*.*

(b) *Since* $\mathrm{SO}(2)$ *is a subgroup of index 2 in* $\mathrm{O}(2)$*, the representation* $E(n)$ *of* $\mathrm{SO}(2) = U(1)$ *induces a representation*

$$\alpha_n = E(n)^{\mathrm{O}(2)}$$

of $\mathrm{O}(2)$ *of degree 2.*

Any element of $\mathrm{O}(2) \setminus \mathrm{SO}(2)$ *is a reflection* $T(l)$ *in some line* l *through the origin. These reflections are all conjugate, since*

$$R(\theta)T(l)R(-\theta) = T(l'),$$

where $l' = R(\theta)l$.
Also

$$T(l)R(\theta)T(l) = R(-\theta);$$

so the $\mathrm{O}(2)$*-conjugacy classes consist of pairs* $\{R(\pm\theta)\}$*, together with the set of all reflections.*
Explicitly, on taking e, Te *as basis for the induced representation (where* T *is any reflection) we see that* α_n *is given by*

$$R(\theta) \mapsto \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad T(l) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0. \end{pmatrix}.$$

If $n \neq 0$ *this representation is simple. For*

$$\alpha_n | \mathrm{SO}(2) = E(n) + E(-n).$$

It follows that the only proper subspaces stable under $\mathrm{SO}(2)$ *are* $\langle e \rangle$, $\langle Te \rangle$*, and these are not stable under* T.
If $n = 0$ *the representation splits into two parts:*

$$\alpha_0 = 1 + \epsilon,$$

where

$$\epsilon(R(\theta)) = 1, \ \epsilon(T(l) = -1,$$

ie $\epsilon(S) = \pm 1$ *according as* S *is proper or improper.*
We claim that the simple representations of $\mathrm{O}(2)$ *are precisely these representations* α_n *for* $n \neq 0$*, together with the representations* $1, \epsilon$ *of degree 1.*
For suppose α *is a simple representation of* $\mathrm{O}(2)$ *in the vector space* V*. Then*

$$\alpha | \mathrm{SO}(2) = E(n_1) + \cdots + E(n_r),$$

ie V *is the direct sum of 1-dimensional subspaces stable under* $\mathrm{SO}(2)$.
Let $U = \langle e \rangle$ *be one such subspace. Then* U *carries some representation* $E(n)$*, ie*

$$R(\theta)e = e^{in\theta}e$$

for all θ.

Take any reflection T. Then the subspace $\langle e, Te\rangle$ is stable under the full group $\mathrm{O}(2)$. Since α is simple,

$$V = \langle e, Te\rangle,$$

If $n \neq 0$ then we see explicitly that

$$\alpha = \alpha_n.$$

If $n = 0$ then $\mathrm{SO}(2)$ acts trivially on U. If $Te = e$ then U is 1-dimensional, and $\alpha = 1$. If not, then the 1-dimensional subspace $\langle e - Te\rangle$ carries the representation ϵ, and so $\alpha = \epsilon$.

We conclude that these are the only simple representations of $\mathrm{O}(2)$.

9. Determine the conjugacy classes in $\mathrm{SU}(2)$.

   Prove that $\mathrm{SU}(2)$ has one simple representation of each dimension $0, 1, 2, \ldots$, and determine the character of this representation.

   Answer:

   (a) *We know that*

   i. *if $U \in \mathrm{SU}(2)$ then U has eigenvalues*

   $$e^{\pm i\theta}\ (\theta \in \mathbb{R}).$$

   ii. *if $X, Y \in \mathrm{GL}(n, k)$ then*

   $$X \sim Y \implies X, Y \text{ have the same eigenvalues.}$$

   A fortiori, if $U \sim V \in \mathrm{SU}(2)$ then U, V have the same eigenvalues.

   We shall show that the converse of the last result is also true, that is: $U \sim V$ in $\mathrm{SU}(2)$ if and only if U, V have the same eigenvalues $e^{\pm i\theta}$, This is equivalent to proving that

   $$U \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

   ie we can find $V \in \mathrm{SU}(2)$ such that

   $$V^{-1}UV = U(\theta).$$

To see this, let v be an $e^{i\theta}$-eigenvalue of U. Normalise v, so that $v^*v = 1$; and let w be a unit vector orthogonal to v, ie $w^*w = 1$, $v^*w = 0$. Then the matrix

$$V = (vw) \in \mathrm{Mat}(2, \mathbb{C})$$

is unitary; and

$$V^{-1}UV = \begin{pmatrix} e^{i\theta} & x \\ 0 & e^{-i\theta} \end{pmatrix}$$

But in a unitary matrix, the squares of the absolute values of each row and column sum to 1. It follows that

$$|e^{i\theta}|^2 + |x|^2 = 1 \implies x = 0,$$

ie

$$V^{-1}UV = U(\theta).$$

We only know that $V \in \mathrm{U}(2)$, not that $V \in \mathrm{SU}(2)$. However

$$V \in \mathrm{U}(2) \implies |\det V| = 1 \implies \det V = e^{i\phi}.$$

Thus

$$V' = e^{-i\phi/2}V \in \mathrm{SU}(2)$$

and still

$$(V')^{-1}UV = U(\theta).$$

To summarise: Since $U(-\theta) \sim U(\theta)$ (by interchange of coordinates), we have show that if

$$C(\theta) = \{U \in \mathrm{SU}(2) : U \text{ has eigenvalues } e^{\pm i\theta}\}$$

then the conjugacy classes in SU(2) *are*

$$C(\theta) \quad (0 \le \theta \le \pi).$$

(b) *Suppose $m \in \mathbb{N}$, Let $V(m)$ denote the space of homogeneous polynomials $P(z, w)$ in z, w. Thus $V(m)$ is a vector space over $\mathbb{C}$ of dimension $m + 1$, with basis $z^m, z^{m-1}w, \ldots, w^m$.*
Suppose $U \in \mathrm{SU}(2)$. Then U acts on z, w by

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathrm{SU}(2)$ *on* $V(m)$*:*

$$P(z, w) \mapsto P(z', w').$$

We claim that the corresponding representation of $\mathrm{SU}(2)$ *— which we denote by* $D_{m/2}$ *— is simple, and that these are the only simple (finite-dimensional) representations of* $\mathrm{SU}(2)$ *over* $\mathbb{C}$.

To prove this, let

$$\mathrm{U}(1) \subset \mathrm{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$*. The action of* $\mathrm{SU}(2)$ *on* z, w *restricts to the action*

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathrm{U}(1)$*. Thus in the action of* $\mathrm{U}(1)$ *on* $V(m)$*,*

$$z^{m-r} w^r \mapsto e^{(m-2r)i\theta} z^{m-r} w^r,$$

It follows that the restriction of $D_{m/1}$ *to* $U(1)$ *is the representation*

$$D_{m/2} | \mathrm{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ *is the representation*

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathrm{U}(1)$*.*

In particular, the character of $D_{m/2}$ *is given by*

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2} i\theta + \cdots + e^{-mi\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$*.*

Now suppose $D_{m/2}$ *is* not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ *is semisimple, since* $\mathrm{SU}(2)$ *is compact.) Let a corresponding split of the representation space be*

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2} | \mathrm{U}(1)$ *are distinct, the expression of* $V(m)$ *as a direct sum of* $\mathrm{U}(1)$*-spaces,*

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1} w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique. It follows that W_1 *must be the direct sum of some of these spaces, and* W_2 *the direct sum of the others. In particular* $z^m \in W_1$ *or* $z^n \in W_2$*, say* $z^m \in W_1$*. Let*

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathrm{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z + w \\ -z + w \end{pmatrix}$$

under U*. Hence*

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r} w^r \rangle$*, it follows that*

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ *of* $\mathrm{SU}(2)$ *in* $V(m)$ *is simple.*

To see that every simple (finite-dimensional) representation of $\mathrm{SU}(2)$ *is of this form, suppose* α *is such a representation. Consider its restriction to* $\mathrm{U}(1)$*. Suppose*

$$\alpha|\,\mathrm{U}(1) = e_r E(r) + e_{r-1} E(r-1) + \cdots + e_{-r} E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α *has character*

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1} e^{(r-1)i\theta} + \cdots + e_{-r} e^{-ri\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$*.*

Since $U(-\theta) \sim U(\theta)$ *it follows that*

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots.$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ *with integer (possibly negative) coefficients:*

$$\chi(\theta) = a_0 \chi_0(\theta) + a_{1/2} \chi_{1/2}(\theta) + \cdots + a_s \chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$