Chapter 1

Compact Groups

> Most infinite groups, in practice, come dressed in a natural *topology*, with respect to which the group operations are *continuous*. All the familiar groups—in particular, all matrix groups—are *locally compact*; and this marks the natural boundary of representation theory.

A *topological group* G is a topological space with a group structure defined on it, such that the group operations

$$(x, y) \mapsto xy, \quad x \mapsto x^{-1}$$

of multiplication and inversion are both continuous.

Examples:

1. The real numbers $\mathbb{R}$ form a topological group under addition, with the usual topology defined by the metric
$$d(x, y) = |x - y|.$$

2. The non-zero reals $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ form a topological group under multiplication, under the same metric.

3. The strictly-positive reals $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$ form a closed subgroup of $\mathbb{R}^\times$, and so constitute a topological group in their own right.

   Remarks:

   (a) Note that in the theory of topological groups, we are only concerned with *closed* subgroups. When we speak of a subgroup of a topological group, it is understood that we mean a closed subgroup, unless the contrary is explicitly stated.

(b) Notice that if a subgroup $H \subset G$ is *open* then it is also closed. For the cosets gH are all open; and so H, as the complement of the union of all other cosets, is closed.

So for example, the subgroup $\mathbb{R}^+ \subset \mathbb{R}^\times$ is both open and closed.

Recall that a space X is said to be *compact* if it is hausdorff and every open covering

$$X = \bigcup_{\imath \in I} U_i$$

has a finite subcovering:

$$X = U_{i_1} \cup \cdots \cup U_{i_r}.$$

(The space X is hausdorff if given any 2 points $x, y \in X$ there exist open sets $U, V \subset X$ such that

$$x \in U, \; y \in V \; U \cap V = \emptyset.$$

All the spaces we meet will be hausdorff; and we will use the term 'space' or 'topological space' henceforth to mean *hausdorff space*.)

In fact all the groups and other spaces we meet will be subspaces of euclidean space E^n. In such a case it is usually easy to determine compactness, since *a subspace $X \subset E^n$ is compact if and only if*

1. *X is closed; and*

2. *X is bounded*

Examples:

1. The *orthogonal group*

$$\mathbf{O}(n) = \{T \in \mathbf{Mat}(n, \mathbb{R}) : T'T = I\}.$$

Here $\mathbf{Mat}(n, \mathbb{R})$ denotes the space of all $n \times n$ real matrices; and T' denotes the transpose of T:

$$T'_{ij} = T_{ji}.$$

We can identify $\mathbf{Mat}(n, \mathbb{R})$ with the Euclidean space E^{n^2}, by regarding the n^2 entries t_{ij} as the *coordinates* of T.

With this understanding, $\mathbf{O}(n)$ is a *closed* subspace of E^{n^2}, since it is the set of 'points' satisfying the simultaneous polynomial equations making up the matrix identity $T'T = I$. It is *bounded* because each entry

$$|t_{ij}| \leq 1.$$

In fact, for each i,

$$t_{1i}^2 + t_{2i}^2 + \cdots + t_{ni}^2 = (T'T)_{ii} = 1.$$

Thus *the orthogonal group $O(n)$ is compact.*

2. The *special orthogonal group*

$$\mathbf{SO}(n) = \{T \in \mathbf{O}(n) : \det T = 1\}$$

is a closed subgroup of the compact group $\mathbf{O}(n)$, and so is itself compact.

Note that
$$T \in \mathbf{O}(n) \Longrightarrow \det T = \pm 1,$$
since
$$T'T = I \Longrightarrow \det T' \det T = 1 \Longrightarrow (\det T)^2 = 1,$$
since $\det T' = \det T$. Thus $\mathbf{O}(n)$ splits into 2 parts: $\mathbf{SO}(n)$ where $\det T = 1$; and a second part where $\det T = -1$. If $\det T = -1$ then it is easy to see that this second part is just the coset $T\mathbf{SO}(n)$ of $\mathbf{SO}(n)$ in $\mathbf{O}(n)$.

We shall find that the groups $\mathbf{SO}(n)$ play a more important part in representation theory than the full orthogonal groups $\mathbf{O}(n)$.

3. The *unitary group*

$$\mathbf{U}(n) = \{T \in \mathbf{Mat}(n, \mathbb{C}) : T^*T = I\}.$$

Here $\mathbf{Mat}(n, \mathbb{C})$ denotes the space of $n \times n$ complex matrices; and T^* denotes the conjugate transpose of T:

$$T^*_{ij} = \overline{T_{ji}}.$$

We can identify $\mathbf{Mat}(n, \mathbb{C})$ with the Euclidean space E^{2n^2}, by regarding the real and imaginary parts of the n^2 entries t_{ij} as the *coordinates* of T.

With this understanding, $\mathbf{U}(n)$ is a *closed* subspace of E^{2n^2}. It is *bounded* because each entry has absolute value

$$|t_{ij}| \le 1.$$

In fact, for each i,

$$|t_{1i}|^2 + |t_{2i}|^2 + \cdots + |t_{ni}|^2 = (T^*T)_{ii} = 1.$$

Thus *the unitary group $\mathbf{U}(n)$ is compact.*

When $n = 1$,

$$\mathbf{U}(1) = \{x \in \mathbb{C} : |x| = 1\}.$$

Thus

$$\mathbf{U}(1) = S^1 \cong \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}.$$

Note that this group (which we can denote equally well by $\mathbf{U}(1)$ or $\mathbb{T}^1$) is *abelian* (or commutative).

4. The *special unitary group*

$$\mathbf{SU}(n) = \{T \in \mathbf{U}(n) : \det T = 1\}$$

is a closed subgroup of the compact group $\mathbf{U}(n)$, and so is itself compact.

Note that

$$T \in \mathbf{U}(n) \Longrightarrow |\det T| = 1.$$

since

$$T^*T = I \Longrightarrow \det T^* \det T = 1 \Longrightarrow |\det T|^2 = 1,$$

since $\det T^* = \overline{\det T}$.

The map

$$U(1) \times \mathbf{SU}(n) \to \mathbf{U}(n) : (\lambda, T) \mapsto \lambda T$$

is a surjective homomorphism. It is not bijective, since

$$\lambda I \in \mathbf{SU}(n) \Longleftrightarrow \lambda^n = 1.$$

Thus the homomorphism has kernel

$$C_n = \langle \omega \rangle,$$

where $\omega = e^{2\pi/n}$. It follows that

$$\mathbf{U}(n) = (\mathbf{U}(1) \times \mathbf{SU}(n))\,/C_n.$$

We shall find that the groups $\mathbf{SU}(n)$ play a more important part in representation theory than the full unitary groups $\mathbf{U}(n)$.

5. The *symplectic group*

$$\mathbf{Sp}(n) = \{T \in \mathbf{Mat}(n, \mathbb{H}) : T^*T = I\}.$$

Here $\mathbf{Mat}(n, \mathbb{H})$ denotes the space of $n \times n$ matrices with *quaternion* entries; and T^* denotes the conjugate transpose of T:

$$T^*_{ij} = \bar{T}_{ji}.$$

(Recall that the conjugate of the quaternion

$$q = t + xi + yj + zk$$

is the quaternion

$$\bar{q} = t - xi - yj - zk.$$

Note that conjugacy is an *anti-automorphism*, ie

$$\overline{q_1 q_2} = \overline{q_2}\,\overline{q_1}.$$

It follows from this that

$$(AB)^* = B^* A^*$$

for any 2 matrices A, B whose product is defined. This in turn justifies our implicit assertion that $Sp(n)$ is a group:

$$S, T \in \mathbf{Sp}(n) \Longrightarrow (ST)^*(ST) = T^*S^*ST = T^*T = I \Longrightarrow ST \in \mathbf{Sp}(n).$$

Note too that while multiplication of quaternions is not in general commutative,

$$\bar{q}q = q\bar{q} = t^2 + x^2 + y^2 + z^2 = |q|^2,$$

defining the norm, or absolute value, $|q|$ of a quaternion q.)

We can identify $\mathbf{Mat}(n, \mathbb{H})$ with the Euclidean space E^{4n^2}, by regarding the coefficients of $1, i, j, k$ in the n^2 entries t_{ij} as the *coordinates* of T.

With this understanding, $\mathbf{Sp}(n)$ is a *closed* subspace of E^{4n^2}. It is *bounded* because each entry has absolute value

$$|t_{ij}| \le 1.$$

In fact, for each i,

$$|t_{1i}|^2 + |t_{2i}|^2 + \cdots + |t_{ni}|^2 = (T^*T)_{ii} = 1.$$

Thus *the symplectic group* $\mathbf{Sp}(n)$ *is compact.*

When $n = 1$,

$$\mathbf{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\} = \{t + xi + yj + zk : t^2 + x^2 + y^2 + z^2 = 1\}.$$

Thus

$$\mathbf{Sp}(1) \cong S^3.$$

We leave it to the reader to show that there is in fact an isomorphism

$$\mathbf{Sp}(1) = \mathbf{SU}(2).$$

Although *compactness* is by far the most important topological property that a group can possess, a second topological property plays a subsidiary but still important rôle—*connectivity*.

Recall that the space X is said to be *disconnected* if it can be partitioned into 2 non-empty open sets:

$$X = U \cup V, \quad U \cap V = \emptyset.$$

We say that X is *connected* if it is not disconnected.

There is a closely related concept which is more intuitively appealing, but is usually more difficult to work with. We say that X is *pathwise-connected* if given any 2 points $x, y \in X$ we can find a path π joining x to y, ie a continuous map

$$\pi : [0,1] \to X$$

with

$$\pi(0) = x, \ \pi(1) = y.$$

It is easy to see that

$$\text{pathwise-connected} \implies \text{connected}.$$

For if $X = U \cup V$ is a disconnection of X, and we choose points $u \in U, v \in V$, then there cannot be a path π joining u to v. If there were, then

$$I = \pi^{-1}U \cup \pi^{-1}V$$

would be a disconnection of the interval $[0,1]$. But it follows from the basic properties of real numbers that the interval is connected. (Suppose $I = U \cup V$. We may suppose that $0 \in U$. Let

$$l = \inf x \in V.$$

Then we get a contradiction whether we assume that $x \in V$ or $x \notin V$.)

Actually, for all the groups we deal with the 2 concepts of*connected* and *pathwise-connected* will coincide. The reason for this is that all our groups will turn out to be *locally euclidean*, ie each point has a neighbourhood homeomorphic to the open ball in some euclidean space E^n. This will become apparent much later when we consider the Lie algebra of a matrix group.

We certainly will not assume this result. We mention it merely to point out that you will not go far wrong if you think of a connected space as one in which you can travel from any point to any other, without ‘taking off’.

The following result provides a useful tool for showing that a compact group is connected.

Proposition 1.1 *Suppose the compact group G acts transitively on the compact space X. Let $x_0 \in X$; and let*

$$H = S(x_0) = \{g \in G : gx_0 = gx_0\}$$

be the corresponding stabiliser subgroup. Then

$$X \text{ connected } \& H \text{ connected } \implies G \text{ connected}.$$

Proof ► By a familiar argument, the action of G on X sets up a 1-1 correspondence between the cosets gH of H in G and the elements of X. In fact, let

$$\Theta : G \to X$$

be the map under which

$$g \mapsto gx_0.$$

Then if $x = gx_0$,

$$\Theta^{-1}\{x\} = gH.$$

Lemma 1.1 *Each coset gH is connected.*

Proof of Lemma ▷ The map

$$h \mapsto gh : H \to gH$$

is a continuous bijection.

But H is compact, since it is a closed subgroup of G (as $H = \Theta^{-1}\{x_0\}$). Now a continuous bijection ϕ of a compact space K onto a hausdorff space Y is necessarily a homeomorphism. For if $U \subset K$ is open, then $C = K \setminus U$ is closed and therefore compact. Hence $\phi(C)$ is compact, and therefore closed; and so $\phi(U) = Y \setminus \phi(C)$ is open in Y. This shows that ϕ^{-1} is continuous, ie ϕ is a homeomorphism.

Thus $H \cong gH$; and so

$$H \text{ connected } \implies gH \text{ connected}.$$

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Now suppose (contrary to what we have to prove) that G is disconnected, say

$$G = U \cup V, \quad U \cap V = \emptyset.$$

This split in G will split each coset:

$$gH = (gH \cap U) \cup (gH \cap V).$$

But hG is connected. Hence

$$gH \subset U \text{ or } gH \subset V.$$

Thus U and V are both unions of cosets; and so under $\Theta : G \to X$ they define a splitting of X:

$$X = \Theta U \cup \Theta V, \quad \Theta U \cap \Theta V = \emptyset.$$

Since U and V are closed (as the complements of each other) and therefore compact, it follows that ΘU and ΘV are compact and therefore closed. Hence each is also open; so X is disconnected.

This is contrary to hypothesis. We conclude that G is connected. ◄

Corollary 1.1 *The special orthogonal group* $\mathbf{SO}(n)$ *is connected for each* n.

Proof ► Consider the action of $\mathbf{SO}(n)$ on $\mathbb{R}^n$:

$$(T, x) \mapsto Tx.$$

This action preserves the norm:

$$\|Tx\| = \|x\|$$

(where $\|x\|^2 = x'x = x_1^2 + \cdots + x_n^2$). For

$$\|Tx\|^2 = (Tx)'Tx = x'T'Tx = x'x.$$

It follows that T sends the sphere

$$S^{n-1} = \{x \in \mathbb{R} : \|x\| = 1\}$$

into itself. Thus $\mathbf{SO}(n)$ acts on S^{n-1}.

This action is transitive: we can find an orthogonal transformation of determinant 1 sending any point of S^{n-1} into any other. (The proof of this is left to the reader.)

Moreover the space S^{n-1} is compact, since it is closed and bounded.

Thus the conditions of our Proposition hold. Let us take

$$x_0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Then

$$H(x_0) = S(x_0) \cong \mathbf{SO}(n-1).$$

For

$$Tx_0 = x_0 \Longrightarrow T = \begin{pmatrix} & & & 0 \\ & T_1 & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}$$

where $T_1 \in \mathbf{SO}(n-1)$. (Since $Tx_0 = x_0$ the last column of T consists of 0's and a 1. But then

$$t_{n1}^2 + t_{n2}^2 + \cdots + 1 = 1 \Longrightarrow t_{n1} = t_{n2} = \cdots = 0.$$

since each row of an orthogonal matrix has norm 1.)

Our proposition shows therefore that

$$\mathbf{SO}(n-1) \text{ connected } \Longrightarrow \mathbf{SO}(n) \text{ connected}.$$

But

$$\mathbf{SO}(1) = \{I\}$$

is certainly connected. We conclude by induction that $\mathbf{SO}(n)$ is connected for all n. ◄

Remark: Although we won't make use of this, our Proposition could be slightly extended, to state that if X is connected, then the number of components of H and G are equal.

Applying this to the full orthogonal groups $\mathbf{O}(n)$, we deduce that for each n $\mathbf{O}(n)$ has the same number of components as $\mathbf{O}(1)$, namely 2. But of course this follows from the connectedness of $\mathbf{SO}(n)$, since we know that $\mathbf{O}(n)$ splits into 2 parts, $\mathbf{SO}(n)$ and a coset of $\mathbf{SO}(n)$ (formed by the orthogonal matrices T with $\det T = -1$) homeomorphic to $\mathbf{SO}(n)$.

Corollary 1.2 *The special unitary group* $\mathbf{SU}(n)$ *is connected for each* n.

Proof ► This follows in exactly the same way. $\mathbf{SU}(n)$ acts on $\mathbb{C}^n$ by

$$(T, x) \mapsto Tx.$$

This again preserves the norm

$$\|x\| = \left(|x_1|^2 + \cdots |x_n|^2\right)^{\frac{1}{2}},$$

since

$$\|Tx\|^2 = (Tx)^* Tx = x^* T^* Tx = x^* x = \|x\|^2.$$

Thus $SU(n)$ sends the sphere

$$S^{2n-1} = \{x \in \mathbb{C}^n : \|x\| = 1\}$$

into itself. As before, the stabiliser subgroup

$$S \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \cong \mathbf{SU}(n-1);$$

and so, again as before,

$$\mathbf{SU}(n-1) \text{ connected } \Longrightarrow \mathbf{SU}(n) \text{ connected.}$$

Since

$$\mathbf{SU}(1) = \{I\}$$

is connected, we conclude by induction that $\mathbf{SU}(n)$ is connected for all n. ◄

Remark: The same argument shows that the full unitary group $\mathbf{U}(n)$ is connected for all n, since

$$U(1) = \{x \in C : |x| = 1\} = S^1$$

is connected.

But this also follows from the connectedness of $\mathbf{SU}(n)$ through the homomorphism

$$(\lambda, T) \mapsto \lambda T : \mathbf{U}(1) \times \mathbf{SU}(n) \to \mathbf{U}(n)$$

since the image of a connected set is connected (as is the product of 2 connected sets).

Note that this homomorphism is not quite an isomorphism, since

$$\lambda I \in \mathbf{SU}(n) \Longleftrightarrow \lambda^n = 1.$$

It follows that

$$\mathbf{U}(n) = (\mathbf{U}(1) \times \mathbf{SU}(n))\,/C_n,$$

where $C_n = \langle\omega\rangle$ is the finite cyclic group generated by $\omega = e^{2\pi/n}$.

Corollary 1.3 *The symplectic group* $\mathbf{Sp}(n)$ *is connected for each* n.

Proof ► The result follows in the same way from the action

$$(T, x) \mapsto Tx$$

of $\mathbf{Sp}(n)$ on $\mathbb{H}^n$. This action sends the sphere

$$S^{4n-1} = \{x \in \mathbb{H}^n : \|x\| = 1\}$$

into itself; and so, as before,

$$\mathbf{Sp}(n-1) \text{ connected } \Longrightarrow \mathbf{Sp}(n) \text{ connected.}$$

In this case we have

$$\mathbf{Sp}(1) = \{q = t + xi + yj + zk \in \mathbb{H} : \|q\|^2 = t^2 + x^2 + y^2 + z^2 = 1\} \cong S^3.$$

So again, the induction starts; and we conclude that $\mathbf{Sp}(n)$ is connected for all n.
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Chapter 2

Invariant integration on a compact group

Every compact group carries a *unique invariant measure*. This remarkable and beautiful result allows us to extend representation theory painlessly from the finite to the compact case.

2.1 Integration on a compact space

There are 2 rival approaches to integration theory.

Firstly, there is what may be called the 'traditional' approach, in which the fundamental notion is the measure $\mu(S)$ of a subset S.

Secondly, there is the 'Bourbaki' approach, in which the fundamental notion is the integral $\int f$ of a function f. This approach is much simpler, where applicable, and is the one that we shall follow.

Suppose X is a compact space. Let $C(X, k)$ (where $k = \mathbb{R}$ or $\mathbb{C}$) denote the vector space of continuous functions

$$f : X \to k.$$

Recall that a continuous function on a compact space is *bounded* and always *attains its bounds*. We set

$$|f| = \max_{x \in X} |f(x)|$$

for each function $f \in C(X, k)$.

This norm defines a metric

$$d(f_1, f_2) = |f_1 - f_2|$$

on $C(X, k)$, which in turn defines a topology on the space.

The metric is *complete*, ie every Cauchy sequence converges. This is easy to see. If $\{f_i\}$ is a Cauchy sequence in $C(X, k)$ then $\{f_i(x)\}$ is a Cauchy sequence in k for each $x \in X$. Since $\mathbb{R}$ and $\mathbb{C}$ are complete metric spaces, this sequence converges, to $f(x)$, say; and it is a simple technical exercise to show that the limit function $f(x)$ is continuous, and that $f_i \mapsto f$ in $C(X, k)$.

Thus $C(X, k)$ is a complete normed vector space—a *Banach space*, in short.

A *measure* μ on X is defined to be *a continuous linear functional*

$$\mu : C(X, k) \to k \qquad (k = \mathbb{R} \text{ or } \mathbb{C}).$$

More fully,

1. μ is *linear*, ie
$$\mu(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \mu(f_1) + \lambda_2 \mu(f_2);$$
2. μ is *continuous*, ie given $\epsilon > 0$ there exists $\delta > 0$ such that
$$|f| < \delta \Longrightarrow |\mu(f)| < \epsilon.$$

We often write

$$\int_X f \, d\mu \text{ or } \int_X f(x) \, d\mu(x)$$

in place of $\mu(f)$.

Since a complex measure μ splits into real and imaginary parts,

$$\mu = \mu_R + i\mu_I,$$

where the measures μ_R and μ_I are real, we can safely restrict the discussion to real measures.

Example: Consider the circle (or torus)

$$S^1 = T = \mathbb{R}/\mathbb{Z}.$$

We parametrise S^1 by the angle $\theta \bmod 2\pi$. The usual measure $d\theta$ is a measure in our sense; in fact

$$\mu(f) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \, d\theta$$

is the invariant Haar measure on the group S^1 whose existence and uniqueness on every compact group we shall shortly demonstrate.

Another measure—a *point* measure—is defined by taking the value of f at a given point, say

$$\mu_1(f) = f(\pi).$$

Measures can evidently be combined linearly, as for example $\mu_2 = \mu + \frac{1}{2}\mu_1$, ie

$$\mu_2(f) = \int_0^{2\pi} f(\theta) \, d\theta + \frac{1}{2} f(\pi).$$

2.2 Integration on a compact group

Suppose now G is a compact group. If μ is a measure on G, and $g \in G$, then we can define a new measure $g\mu$ by

$$(g\mu)(f) = \mu(g^{-1}f) = \int_G f(gx)\, dg.$$

(Since we are dealing with functions on a space of functions, g is inverted twice.)

Theorem 2.1 *Suppose G is a compact group. Then there exists a unique real measure μ on G such that*

1. *μ is* invariant *on G, ie*

$$\int_G (gf)\, d\mu = \int_G f\, d\mu$$

   for all $g \in G, f \in C(G, \mathbb{R})$.

2. *μ is* normalised *so that G has volume 1, ie*

$$\int_G 1\, d\mu = 1.$$

Moreover,

1. *this measure is* strictly positive, *ie*

$$f(x) \geq 0 \text{ for all } x \implies \int f\, d\mu \geq 0,$$

   with equality only if $f = 0$, ie $f(g) = 0$ for all g.

2.
$$|\int_G f\, d\mu| \leq |f|.$$

Proof ►

The intuitive idea. As the proof is long, and rather technical, it may help to sketch the argument first. The basic idea is that *averaging smoothes*.

By an *average* $F(x)$ of a function $f(x) \in C(G)$ we mean *a weighted average of transforms of f*, ie a function of the form

$$F(x) = \lambda_1 f(g_1 x) + \cdots + \lambda_r f(g_r x),$$

where

$$g_1, \ldots, g_r \in G,\ 0 \leq \lambda_1, \ldots, \lambda_r \leq 1,\ \lambda_1 + \cdots + \lambda_r = 1.$$

These averages have the following properties:

- An average of an average is an average, ie if F is an average of f, then an average of F is also an average of f.
- If there *is* an invariant measure on F, then averaging leaves the integral unchanged, ie if F is an average of f then

$$\int F \, dg = \int f \, dg.$$

- Averaging smoothes, in the sense that if F is an average of f then

$$\min f \leq \min F \leq \max F \leq \max f.$$

  In particular, if we define the *variation* of f by

$$\operatorname{var} f = \max f - \min f$$

  then

$$\operatorname{var} F \leq \operatorname{var} f.$$

Now suppose a positive invariant measure exists. Then

$$\min f \leq \int f \, dg \leq \max f,$$

ie the integral of f is sandwiched between its bounds.

Thus if we can find an average $F(x)$ with small $\operatorname{var}(F)$ then this will give us a good estimate for

$$\int f dg = \int F dg.$$

If f is not completely smooth, ie constant, we can always make it smoother, ie reduce its variation, by 'filling its valleys from its mountains', as follows. Let

$$m = \min f, \quad M = \max f;$$

and let U be the set of points where f is 'below average', ie

$$U = \{x \in G : f(x) < \frac{1}{2}(m + M)\}.$$

The transforms of U (as of any non-empty set) cover X. (For if $x \in G$ and $x_0 \in U$ then $x \in (x x_0^{-1})U$.) Since U is open, and X is compact, a finite number of these transforms cover X, say

$$X \subset g_1 U \cup \cdots \cup g_n U.$$

Now consider the average

$$F = \frac{1}{n}\left(g_1 f + \cdots + g_n f\right),$$

ie

$$F(x) = \frac{1}{n}\Big(f(g_1^{-1}x) + \cdots + f(g_n^{-1}x)\Big).$$

For any x, at least one of $g_1^{-1}x, \dots, g_r^{-1}x$ lies in U (since $x \in g_i U \implies g_i^{-1}x \in U$). Hence

$$\begin{aligned} F(x) &< \frac{1}{n}\Big((n-1)M + \frac{1}{2}(m+M)\Big) \\ &= \Big(1 - \frac{1}{2n}M\Big) + \frac{1}{2n}m. \end{aligned}$$

Thus

$$\operatorname{var} F < \Big(1 - \frac{1}{2n}\Big)(M-m) < \operatorname{var} f.$$

Unfortunately, while this argument proves that $\operatorname{var}(F)$ can be reduced, it does not show that we can make $\operatorname{var}(F) \to 0$. For that, we need to use the fact that *continuity on a compact group implies uniform continuity*.

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be uniformly continuous on the interval $I \subset \mathbb{R}$ if given $\epsilon > 0$ we can always find $\delta > 0$ such that

$$|x - y| < \delta \implies |fx - fy| < \epsilon.$$

We can extend this concept to a function $f : G \to \mathbb{R}$ on a compact group G as follows: f is said to be uniformly continuous on G if given $\epsilon > 0$ we can find an open set $U \ni 1$ (the neutral element of G) such that

$$x^{-1}y \in U \implies |fx - fy| < \epsilon.$$

Lemma 2.1 *A continuous function on a compact group is necessarily uniformly continuous.*

Proof of Lemma $\triangleright$ Suppose $f \in C(G, k)$. For each point $g \in G$, let

$$U(g) = \{x \in G : |f(x) - f(g)| < \frac{1}{2}\epsilon\}.$$

By the triangle inequality,

$$x, y \in U(g) \implies |f(x) - f(y)| < \epsilon.$$

Now each neighbourhood U of g in G is expressible in the form

$$U = gV$$

where V is a neighbourhood of the neutral element 1.

Furthermore, for each neighbourhood V of e, we can find a smaller neighbourhood W of e such that

$$W^2 \subset V.$$

(This follows from the continuity of the multiplication $(x, y) \mapsto xy$. Here W^2 denotes the set $\{w_1 w_2 : w_1, w_2 \in W\}$.)

So for each $g \in G$ we can find an open neighbourhood $W(g)$ of e such that

$$gW(g)^2 \subset U(g);$$

and in particular

$$x, y \in gW(g)^2 \Longrightarrow |f(x) - f(y)| < \epsilon.$$

The open sets $gW(g)$ cover G (since $g \in W(g)$). Therefore, since G is compact, we can find a finite subcover, say

$$G = g_1 W_1 \cup g_2 W_2 \cup \cdots \cup g_n W_n,$$

where $W_i = W(g_i)$.

Let

$$W = \cap_i W_i.$$

Suppose $x^{-1}y \in W$, ie

$$y \in xW.$$

Then x lies in some set $g_i W_i$. Thus

$$x, y \in g_i W_i W \subset g_i W_i^2;$$

and so

$$|f(x) - f(y)| < \epsilon.$$

$\triangleleft$

Proof proper:

Lemma 2.2 *Suppose $f \in C(G, \mathbb{R})$. Then there exists an average F of f with*

$$\operatorname{var}(F) \leq \frac{1}{2} \operatorname{var} f.$$

Proof of Lemma ▷ Suppose $\epsilon > 0$. (We shall choose ϵ later.) By the previous Lemma $f(x)$ is uniformly continuous; so we can find an open set $U \ni 1$ such that

$$y^{-1}x \in U \implies |f(y) - f(x)| < \epsilon.$$

The result holds — with the same U — for each transform $f(g^{-1}x)$, since

$$(g^{-1}y)^{-1}(g^{-1}x) = y^{-1}x.$$

It follows that *the same result holds for each average*

$$F(x) = \lambda_1 f(h_1^{-1}x) + \cdots + \lambda_n f(h_n^{-1}x).$$

Since G is compact, we can find transforms of U such that

$$G = g_1 U \cup \cdots \cup g_n U.$$

Suppose $f(x)$ attains its lower bound m at x_0. Let

$$x_0 \in g_0 U,$$

where $g_0 \in \{g_1, \ldots, g_n\}$.

Now suppose x is a general point of G. Let

$$x \in g_i U.$$

Then

$$g_0 g_i^{-1} x \in g_0 U, \ x_0 \in g_0 U.$$

It follows that

$$|f(g_0 g_i^{-1} x) - f(x_0)| < 2\epsilon.$$

(For both $g_0 g_i^{-1} x$ and x_0 are U-close to g_0.) But $f(x_0) = m$. Thus

$$f(g_0 g_i^{-1} x) < m + 2\epsilon.$$

Now consider the average

$$F(x) = \frac{1}{n^2} \sum_{1 \le i,j \le n} f(g_j g_i^{-1} x).$$

We have

$$F(x) < \frac{n^1 - 1}{n^2} M + \frac{1}{n^2}(m + 2\epsilon).$$

Similarly — considering the point where $f(x)$ attains its maximum M rather than its minimum m —

$$F(x) > \frac{n^1 - 1}{n^2} m + \frac{1}{n^2}(M - 2\epsilon).$$

Thus

$$\operatorname{var}(F) < \left(1 - \frac{2}{n^2}\right) \operatorname{var}(f) + \frac{4}{n^2}\epsilon.$$

Now we can repeat the same argument with F in place of f (but with the same U and n).

$\triangleleft$

By the argument above, we can find a sequence of averages

$$F_0 = f, F_1, F_2, \ldots$$

(each an average of its predecessor) such that

$$\operatorname{var} F_0 > \operatorname{var} F_1 > \operatorname{var} F_2 > \cdots$$

(or else we reach a constant function $F_r = c$).

However, this does not establish that

$$\operatorname{var} F_i \to 0$$

as $i \to \infty$. We need a slightly sharper argument to prove this. In effect we must use the fact that f is *uniformly continuous*. Now we observe that this open set U will serve not only for f but also for every average F of f. For if

$$F = \lambda_1 g_1 f + \cdots + \lambda_r g_r f \quad (0 \leq \lambda_1, \ldots, \lambda_r \leq 1,\ \lambda_1 + \cdots + \lambda_r = 1)$$

then

$$|F(x) - F(y)| \leq \lambda_1 |f(g_1^{-1}x) - f(g_1^{-1}y)| + \cdots + \lambda_r |f(g_r^{-1}x) - f(g_r^{-1}y)|.$$

But

$$(g_i^{-1}x)^{-1}(g_i^{-1}y) = x^{-1}g_i g_i^{-1}y = x^{-1}y.$$

Thus

$$\begin{aligned} x^{-1}y \in U &\Longrightarrow |f(g_i^{-1}x) - f(g_i^{-1}y)| < \epsilon \\ &\Longrightarrow |F(x) - F(y)| < (\lambda_1 + \cdots \lambda_r)\epsilon = \epsilon. \end{aligned}$$

Returning to our construction of an 'improving average' F, let us take $\epsilon = (M - m)/4$; then we can find an open set $U \ni 1$ such that

$$x^{-1}y \in U \Longrightarrow |F(x) - F(y)| < \frac{1}{2}(M - m)$$

for every average F of f. It follows that the variation of F on any transform gU is less than half the variation of f on G.

As before, we can find a finite number of transforms of U covering G, say

$$G \subset g_1 U \cup \cdots \cup g_r U.$$

Suppose $f(x)$ attains its minimum m at some point x_0. Let

$$x_0 \in g_0 U.$$

Now take a general point $x \in G$. Suppose

$$x \in g_i U.$$

Then

$$g_0 g_i^{-1} x \in g_0 U \text{ and } x_0 \in g_0 U.$$

Hence

$$|f(g_0 g_i^{-1} x - x_0| < (M - m)/2;$$

and so (since $f(x_0) = m$)

$$f(g_0 g_i^{-1} x < m + (M - m)/2 = (M + m)/2$$

In other words, the value of the function at the point $g_0 g_i^{-1}$ is ‘below the half-way mark’.

Now let us take the average

$$F(x) = \frac{1}{n^2} \sum i, j f(g_i g_j^{-1} x).$$

We have

$$\begin{aligned} F(x) &< \frac{n^2 - 1}{n^2} M + \frac{1}{n^2}(M + m)/2 \\ &= M - \frac{1}{2n^2}(M - m). \end{aligned}$$

Thus

$$\sup(F) \leq sup(f) - \frac{1}{2n^2} \operatorname{var}(f).$$

Similarly

$$\inf(F) \geq inf(f) + \frac{1}{2n^2} \operatorname{var}(f).$$

Hence

$$\operatorname{var} F \leq \frac{n^2 - 1}{n^2} \operatorname{var}(f).$$

At first sight, this seems a weaker result than our earlier one, which showed that $\operatorname{var} F' < \operatorname{var} F$ in *all* cases! The difference is, that r now is *independent of* F. Thus we can find a sequence of averages

$$F_0 = f, F_1, F_2, \ldots$$

(each an average of its predecessor) such that $\operatorname{var} F_i$ is decreasing to a limit ℓ satisfying

$$\ell \le \left(1 - \frac{1}{r}\right)\ell + \frac{1}{2r}\operatorname{var}(f),$$

ie

$$\ell \le \frac{1}{2}\operatorname{var} f.$$

In particular, we can find an average F with

$$\operatorname{var} F < \frac{2}{3}\operatorname{var} f.$$

Repeating the argument, with F in place of f, we find a second average F' such that

$$\operatorname{var} F' < \left(\frac{2}{3}\right)^2 \operatorname{var} f;$$

and further repetition gives a new sequence of averages

$$F_0 = f, F_1, F_2, \ldots,$$

with

$$\operatorname{var} F_i \to 0,$$

as required.

This sequence gives us a nest of intervals

$$(\min f, \max f) \supset (\min F_1, \max F_1) \supset (\min F_2, \max F_2) \cdots$$

whose lengths are tending to 0. Thus the intervals converge on a unique real number I.

We want to set

$$\int f\, dg = I.$$

But before we can do this, we must ensure that no other sequence of averages can lead to a nest of intervals

$$(\min f, \max f) \supset (\min F_1', \max F_1') \supset (\min F_2', \max F_2') \cdots$$

converging on a different real number $I' \ne I$.

This will follow at once from the following Lemma.

Lemma 2.3 *Suppose F, F' are two averages of f. Then*

$$\min F \leq \max F'.$$

In other words, the minimum of any average is $\leq$ the maximum of any other average.

Proof of Lemma $\triangleright$ The result would certainly hold if we could find a function F'' which was an average both of F and of F'; for then

$$\min F \leq \min F'' \leq \max F'' \leq maxF'.$$

However, it is not at all clear that such a 'common average' always exists. We need a new idea.

So far we have only been considering the action of G on $C(G)$ *on the left*. But G also acts on the right, the 2 actions being independent and combining in the action of $G \times G$ given by

$$((g, h)f)(x) = f(g^{-1}xh).$$

Let us temporarily adopt the notation fh for this right action, ie

$$(fh)(x) = f(xh).$$

We can use this action to define *right averages*

$$\sum \mu_j(fh_j).$$

The point of introducing this complication is that we can use the right averages to *refine* the left averages, and vice versa.

Thus suppose we have a left average

$$F = \sum \lambda_i(g_i f)$$

and a right average

$$F' = \sum \mu_j(fh_j).$$

Then we can form the *joint average*

$$F'' = \sum \sum \lambda_i \mu_j (g_i f h_j).$$

We can regard F'' as arising either from F by right-averaging, or from F' by left averaging. In either case we conclude that F'' is 'smoother' (ie has smaller variation) than either F of F'; and

$$\min F \leq \min F'' \leq \max F'' \leq \max F'.$$

Thus the minimum of any left average is $\leq$ the maximum of any right average. Similarly

$$\min F' \leq \max F;$$

the minimum of any right average is $\leq$ the maximum of any left average.

In fact, the second result follows from the first; since we can pass from left averages to right averages, and vice versa, through the involution

$$f \to \tilde{f} : C(G) \to C(G),$$

where

$$\tilde{f}(g) = f(g^{-1}).$$

For it is readily verified that

$$F = \lambda_1(g_1 f) + \cdots + \lambda_r(g_r f) \Longrightarrow \tilde{F} = \lambda_1(\tilde{f} g_1^{-1}) + \cdots + \lambda_r(\tilde{f} g_r^{-1}).$$

Thus if F is a left average then $\tilde{F}$ is a right average, and vice versa.

Now suppose we have 2 left averages F_1, F_2 such that

$$\max F_1 < \min F_2.$$

Let

$$\min F_2 - \max F_1 = \epsilon.$$

Let F' be a right average with

$$\operatorname{var} F' = \max F' - \min F' < \epsilon.$$

Then we have a contradiction; for

$$\min F_2 \leq \max F' < \min F' + \epsilon \leq \max F_1 + \epsilon.$$

$\triangleleft$

We have shown therefore that there is no ambiguity in setting

$$\mu(f) = I,$$

where I is the limit of a sequence of averages $F_0 = f, F_1, \ldots$ with $\operatorname{var} F_i \to 0$; for any two such sequences must converge to the same value.

It remains to show that this defines a *continuous* and *linear* function

$$\mu : C(G, \mathbb{R}) \to \mathbb{R}.$$

Let us consider linearity first. It is evident that

$$\mu(\lambda f) = \lambda \mu(f),$$

since multiplying f by a scalar will multiply all averages by the same number.

Suppose $f_1, f_2 \in C(G, \mathbb{R})$. Our argument above showed that the right averages of f converge on the same constant value $\mu(f) = I$. So now we can take a left average of f_1 and a right average of f_2, and add them to give an average of $f_1 + f_2$. More precisely, given $\epsilon > 0$ we can find a left average

$$F_1 = \sum \lambda_i g_i f_1$$

of f_1 such that

$$\mu(f_1) - \epsilon < \min F_1 \leq \max F_1 < \mu(f_1) + \epsilon;$$

and similarly we can find a right average

$$F_2 = \sum \mu_j f_2 h_j$$

of f_2 such that

$$\mu(f_2) - \epsilon < \min F_2 \leq \max F_2 < \mu(f_2) + \epsilon.$$

Now let

$$F = \sum_i \sum_j \lambda_i \mu_j \left(g_i (f_1 + f_2) h_j \right).$$

Then we have

$$\min F_1 + \min F_2 \leq \min F \leq \mu(f + g) \leq \max F \leq \max F_1 + \max F_2;$$

from which we deduce that

$$\mu(f + g) = \mu(f) + \mu(g).$$

Let's postpone for a moment the proof that μ is continuous.

It is evident that a non-negative function will have non-negative integral, since all its averages will be non-negative:

$$f \geq 0 \Longrightarrow \int f \, d\mu \geq 0.$$

It's perhaps not obvious that the integral is *strictly* positive. Suppose $f \geq 0$, and $f(g) > 0$. Then we can find an open set U containing g such that

$$f(x) \geq \delta > 0$$

for $x \in U$. Now we can find $g_1, \ldots, g_r$ such that

$$G = g_1 U \cup \cdots \cup g_r U.$$

Let F be the average

$$F(x) = \frac{1}{r}\left(f(g_1^{-1}x) + \cdots + f(g_r^{-1}x)\right).$$

Then

$$x \in g_iU \Longrightarrow g_i^{-1}x \in U \Longrightarrow f(g_i^{-1}x) \geq \delta,$$

and so

$$F(x) \geq \frac{\delta}{r}.$$

Hence

$$\int f\ dg = \int F\ dg \geq \frac{\delta}{r} > 0.$$

Since

$$\min f \leq \int f\ d\mu \leq \max f,$$

it follows at once that

$$|\int f\ d\mu| \leq |f|.$$

It is now easy to show that μ is continuous. For a linear function is continuous if it is continuous at 0; and we have just seen that

$$|f| < \epsilon \Longrightarrow |\int f\ d\mu| < \epsilon.$$

It follows at once from

$$\min f \leq \int f\ dg \leq \max f$$

that

$$|\int f\ dg| \leq |f|.$$

Finally, since f and gf (for $f \in C(G),\ g \in G$) have the same transforms, they have the same (left) averages. Hence

$$\int gf\ dg = \int f\ dg,$$

ie the integral is left-invariant.

Moreover, it follows from our construction that this is the only left-invariant integral on G with $\int 1\ dg = 1$; for any such integral must be sandwiched between $\min F$ and $\max F$ for all averages F of f, and we have seen that these intervals converge on a single real number. ◄

The Haar measure, by definition, is *left* invariant:

$$\int f(g^{-1}x)\ d\mu(x) = \int f(x)\ d\mu(x).$$

It followed from our construction that it is also *right* invariant:

$$\int f(xh)\, d\mu(x) = \int f(x)\, d\mu(x).$$

It is worth noting that this can be deduced directly from the existence of the Haar measure.

Proposition 2.1 *The Haar measure on a compact group G is right invariant, ie*

$$\int_G f(gh)\, dg = \int_G f(g)\, dg \qquad (h \in G, f \in C(G, \mathbb{R}))\,.$$

Proof ► Suppose $h \in G$. The map

$$\mu_h : f \mapsto \mu(fh)$$

defines a left invariant measure on G. By the uniqueness of the Haar measure, and the fact that

$$\mu_h(1) = 1$$

(since the constant function 1 is right as well as left invariant),

$$\mu_h = \mu,$$

ie μ is right invariant. ◄

Outline of an alternative proof Those who are fond of abstraction might prefer the following formulation of the first part of our proof, set in the real Banach space $C(G) = C(G, \mathbb{R})$.

Let $\mathcal{A}(f) \subset C(G)$ denote the set of averages of f. This set is *convex*, ie

$$F, F' \in \mathcal{A}(f) \Longrightarrow \lambda F + (1-\lambda)F' \in \mathcal{A}(f) \quad (0 \le \lambda \le 1).$$

Let $\Lambda \subset C(G)$ denote the set of constant functions $f(g) = c$. Evidently

$$\Lambda \cong \mathbb{R}.$$

We want to show that

$$\Lambda \cap \overline{\mathcal{A}(f)} \neq \emptyset,$$

ie the closure of $\mathcal{A}(f)$ contains a constant function. (In other words, we can find a sequence of averages converging on a constant function.)

To prove this, we establish that $\mathcal{A}(f)$ is *pre-compact*, ie its closure $\overline{\mathcal{A}(f)}$ is compact. For then it will follow that there is a 'point' $X \in \overline{\mathcal{A}(f)}$ (ie a function $X(g)$) which is *closest* to Λ. But if this point is not in Λ, we will reach a contradiction; for by the same argument that we used in our proof, we can always improve on a non-constant average, ie find another average closer to Λ. (We actually need the stronger version of this using uniform continuity, since the 'closest point' $X(g)$ is not necessarily an average, but only the limit of a sequence of averages. Uniform continuity shows that we can improve all averages by a fixed amount; so if we take an average sufficiently close to $X(g)$ we can find another average closer to Λ than $X(g)$.)

It remains to show that $\mathcal{A}(f)$ is pre-compact. We note in the first place that the set of transforms of f,

$$Gf = \{gf : g \in G\}$$

is a compact subset of $C(G)$, since it is the image of the compact set G under the continuous map

$$g \mapsto gf : G \to C(G).$$

Also, $\mathcal{A}(f)$ is the *convex closure* of this set Gf, ie the smallest convex set containing Gf (eg the intersection of all convex sets containing G), formed by the points

$$\{\lambda_1 F_1 + \cdots + \lambda_r F_r : 0 \leq \lambda_1, \ldots, \lambda_r \leq 1; \lambda_1 + \cdots + \lambda_r = 1\}.$$

Thus $\mathcal{A}(f)$ is the convex closure of the compact set Gf. But the convex closure of a compact set in a complete metric space is always pre-compact. That follows (not immediately, but by a straightforward argument) from the following lemma in the theory of metric spaces: *A subset $S \subset X$ of a complete metric space is pre-compact if and only if it can be convered by a finite number of balls of radius ϵ,*

$$S \subset B(x_1, \epsilon) \cup \cdots \cup B(x_r, \epsilon),$$

for every $\epsilon > 0$.

Accordingly, we have shown that $\Lambda \cap \overline{\mathcal{A}(f)}$ is non-empty. We must then show that it consists of a single point. This we do as in our proof proper, by introducing right averages. Finally, we define $\int f \; dg$ to be this point of intersection (or rather, the corresponding real number); and we show as before that this defines an invariant integral $\mu(f)$ with the required properties.

Examples:

1. As we have already noted, the Haar measure on S^1 is

$$\frac{1}{2\pi} d\theta.$$

In other words,

$$\mu(f) = \frac{1}{2\pi} \int_0^2 \pi f(\theta) \; d\theta.$$

2. Consider the compact group $\mathbf{SU}(2)$. We know that

$$\mathbf{SU}(2) \cong S^3,$$

since the general matrix in $\mathbf{SU}(2)$ takes the form

$$U = \begin{pmatrix} x+iy & z+it \\ -z+it & x-iy \end{pmatrix}, \quad |x|^2+|y|^2+|z|^2+|t|^2 = 1.$$

The usual volume on S^3, when normalised, gives the Haar measure on $SU(2)$. To see that, observe that multiplication by $U \in SU(2)$ defines a distance preserving linear transformation—an *isometry*—of $\mathbb{R}^4$, ie if

$$U \begin{pmatrix} x+iy & z+it \\ -z+it & x-iy \end{pmatrix} = \begin{pmatrix} x'+iy' & z'+it' \\ -z'+it' & x'-iy' \end{pmatrix}$$

then

$$x'^2 + y'^2 + z'^2 + t'^2 = x^2 + y^2 + z^2 + t^2$$

for all $(x, y, z, t) \in \mathbb{R}^4$.

It follows that multiplication by U preserves the volume on S^3. In other words, this volume provides an invariant measure on $\mathbf{SU}(2)$, which must therefore be—after normalisation—the Haar measure on $\mathbf{SU}(2)$.

As this example—the simplest non-abelian compact group—demonstrates, concrete computation of the Haar measure is likely to be complicated. Fortunately, the mere *existence* of the Haar measure is usually sufficient for our purpose.

Chapter 3

From finite to compact groups

Almost all the results established in Part I for finite-dimensional representations of *finite* groups extend to finite-dimensional representations of *compact* groups. For the Haar measure on a compact group G allows us to *average* over G; and our main results were—or can be—established by averaging.

In this chapter we run very rapidly over these results, and their extension to the compact case. This may serve (if nothing else) as a review of the main results of finite-dimensional representation theory.

The chapter is divided into sections corresponding to the chapters of Part I, eg section 3.5 covers the results established in chapter 5 of Part I.

We assume, unless the contrary is explicitly stated, that we are dealing with *finite-dimensional* representations over k (where $k = \mathbb{R}$ or $\mathbb{C}$). This restriction greatly simplifies the story, for three reasons:

1. Each finite-dimensional vector space over k carries a *unique* hausdorff topology under which addition and scalar multiplication are continuous. If V is n-dimensional then
$$V \cong k^n;$$
and this unique topology on V is just that arising from the product topology on k^n.

2. If U and V are finite-dimensional vector spaces over k, then every linear map
$$t : U \to V$$
is continuous. *Continuity is automatic in finite dimensions.*

3. If V is a finite-dimensional vector space over k, then every subspace $U \subset V$ is *closed* in V.

3.1 Representations of a Compact Group

We have agreed that a representation of a topological group G in a finite-dimensional vector space V over k (where $k = \mathbb{R}$ or $\mathbb{C}$) is defined by a *continuous linear action*

$$G \times V \to V.$$

Recall that a representation of a finite group G in V can be defined in 2 equivalent ways:

1. by a linear action
$$G \times V \to V;$$
2. by a homomorphism
$$G \to \mathbf{GL}(V),$$
where $\mathbf{GL}(V)$ denotes the group of invertible linear maps $t : V \to V$.

We again have the same choice. We have chosen (1) as our fundamental definition in the compact case, where we chose (2) in the finite case, simply because it is a little easier to discuss the continuity of a linear action.

However, there is a natural topology on $\mathbf{GL}(V)$. For we can identify $\mathbf{GL}(V)$ with a subspace of the space of *all* linear maps $t : V \to V$; if $\dim V = n$ then

$$\mathbf{GL}(V) \subset \mathbf{Mat}(n, k) \cong k^{n^2}.$$

This n^2-dimensional vector space has a unique hausdorff topology, as we have seen; and this induces a topology on $\mathbf{GL}(V)$.

We know that there is a one-one correspondence between linear actions of G on V and homomorphisms $G \to \mathbf{GL}(V)$. It is a straightforward matter to verify that under this correspondence, *a linear action is continuous if and only if the corresponding homomorphism is continuous*.

3.2 Equivalent Representations

The definition of the equivalence of 2 representations α, β of a group G in the finite-dimensional vector spaces U, V over k holds for *all* groups, and so extends without question to compact groups.

We note that the map $\theta : U \to V$ defining such an equivalence is necessarily continuous, since U and V are finite-dimensional. In the infinite-dimensional case (which, we emphasise, we are not considering at the moment) we would have to *add* the requirement that θ should be continuous.

3.3 Simple Representations

Recall that the representation α of a group G in the finite-dimensional vector space V over k is said to be *simple* if no proper subspace $U \subset V$ is stable under G. This definition extends to all groups G, and in particular to compact groups.

In the infinite-dimensional case we would restrict the requirement to proper *closed* subspaces of V. This is no restriction in our case, since as we have noted, all subspaces of a finite-dimensional vector space over k are closed.

3.4 The Arithmetic of Representations

Suppose α, β are representations of the group G in the finite-dimensional vector spaces U, V over k. We have defined the representations $\alpha + \beta$, $\alpha\beta$, α^* in the vector spaces $U \oplus V$, $U \otimes V$, U^*, respectively. These definitions hold for all groups G.

However, there *is* something to verify in the topological case, even if it is entirely straightforward. We must show that if α and β are continuous then so are $\alpha + \beta$, $\alpha\beta$, and α^*. (This is left as an exercise to the student.)

3.5 Semisimple Representations

The definition of the semisimplicity of a representation α of a group G in a finite-dimensional vector space V over k makes no restriction on G, and so extends to compact groups (and indeed to all topological groups); α is semisimple if and only if it is expressible as a sum of simple representations:

$$\alpha = \sigma_1 + \cdots + \sigma_m.$$

Recall that a finite-dimensional representation of G in V is semisimple if and only if each stable subspace $U \subset V$ has at least one stable complementary subspace $W \subset V$:

$$V = U \oplus W.$$

We shall see later that this provides us with a definition of semisimplicity which extends easily to infinite-dimensional representations,

3.6 Every Representation of a Finite Group is Semisimple

This result is the foundation-stone of our theory; and its extension from finite to compact groups is a triumph for Haar measure.

Let us imitate our first proof of the result in the finite case. Suppose α is a representation of G in the finite-dimensional vector space V over k (where $k = \mathbb{R}$ or $\mathbb{C}$).

Recall that we start by taking any positive-definite inner product (quadratic if $k = \mathbb{R}$, hermitian if $k = \mathbb{C}$) $P(u, v)$ on V. Next we *average* P over G, to give a new inner product

$$\langle u, v\rangle = \int_V p(gu, gv)\ dg.$$

It is a straightforward matter to verify that this new inner product is invariant:

$$\langle gu, gv\rangle = \langle u, v\rangle.$$

It also follows at once from the positivity of the Haar measure that this inner product is positive, ie

$$\langle v, v\rangle \geq 0.$$

It's a little more difficult to see that the inner product is positive-*definite*, ie

$$\langle v, v\rangle = 0 \Longrightarrow v = 0.$$

However, this follows at once from the fact that the Haar measure on a compact group is itself *positive-definite*, in the sense that if $f(g)$ is a continuous function on G such that $f(g) \geq 0$ for all $g \in G$ then not only is

$$\int_G f(g)\ dg \geq 0$$

(this is the positivity of the measure) but also

$$\int_G f(g)\ dg = 0 \Longrightarrow f(g) = 0 \text{ for all } g.$$

This follows easily enough from the fact that if $f(g_0) = \epsilon > 0$, then $f(g) \geq \epsilon/2$ for all $g \in U$ where U is an open neighbourhood of g_0. But then (since G is compact) G can be covered by a finite number of transforms of U:

$$G \subset g_1U \cup \ldots g_rU.$$

It follows from this that

$$f(g_1^{-1}x) + \cdots + f(g_r^{-1}x) \geq \epsilon/2$$

for all $x \in G$. For

$$x \in g_i U \Longrightarrow g_i^{-1} x \in U \Longrightarrow f(g_i^{-1} x) \geq \epsilon/2.$$

It follows from this, on integrating, that

$$r \int_G f(g)\, dg \geq \epsilon/2.$$

In particular $\int f \geq 0$.

Note that our alternative proof of semisimplicity also carries over to the compact case. This proof depended on the fact that if

$$\pi : V \to V$$

is a *projection* onto a *stable* subspace $U = \pi(V)$ of V then its *average*

$$\Pi = \frac{1}{|G|} \sum_{g \in G} g \pi g^{-1}$$

is also a projection onto U; and

$$W = \ker \Pi$$

is a stable complementary subspace:

$$V = U \oplus W.$$

This carries over without difficulty, although a little care is required. First we must explain how we define the average

$$\Pi = \int_G g \pi g^{-1}\, dg.$$

For here we are integrating the *operator-valued* function

$$F(g) = g \pi g^{-1}$$

However, there is little difficulty in extending the concept of measure to *vector-valued* functions F on G, ie maps

$$F : G \to V,$$

where V is a finite-dimensional vector space over k. This we can do, for example, by choosing a basis for V, and integrating each component of F separately. We must show that the result is independent of the choice of basis; but that is straightforward, The case of a function with values in $\hom(U, V)$, where U, V are finite-dimensional vector spaces over k, may be regarded as a particular case of this, since we can regard $\hom(U, V)$ as itself a vector space over k.

There is one other point that arises: in this proof (and elsewhere) we often encounter double sums

$$\sum_{g \in G} \sum_{h \in G} f(g, h)$$

over G. The easiest way to extend such an argument to compact groups is to consider the corresponding integral

$$\int_{G \times G} f(g, h)\; d(g, h)$$

of the continuous function $f(g, h)$ over the product group $G \times G$.

In such a case, let us set

$$F(g) = \int_{h \in G} f(g, h)\; dh$$

for each $g \in G$. Then it is readily shown that $F(g)$ is continuous, so that we can compute

$$I = \int_{g \in G} F(g)\; dg$$

But then it is not hard to see that $I = I(f)$ defines a second Haar measure on $G \times G$; so we deduce from the uniqueness of this measure that

$$\int_{G \times G} f(g, h)\; d(g, h) = \int_{g \in G} \left(\int_{h \in G} f(g, h)\; dh \right)\; dg.$$

This result allows us to deal with all the manipulations that arise (such as reversal of the order of integration). For example, in our proof of the result above that the averaged projection Π is itself a projection, we argue as follows:

$$\begin{aligned} \Pi^2 &= \int_{g \in G} g\pi g^{-1}\; dg \int_{h \in G} h\pi h^{-1}\; dh \\ &= \int_{(g,h) \in G \times G} g\pi g^{-1} h\pi h^{-1}\; d(g, h) \\ &= \int_{(g,h) \in G \times G} g g^{-1} h\pi h^{-1}\; d(g, h) \end{aligned}$$

(using the fact that $\pi g \pi = g\pi$, since $U = \operatorname{im} \pi$ is stable under G). Thus

$$\begin{aligned} \Pi^2 &= \int_{(g,h) \in G \times G} h\pi h^{-1}\; d(g, h) \\ &= \int_{g \in G} dg \int_{h \in G} h\pi h^{-1}\; dh \\ &= \Pi. \end{aligned}$$

3.7 Uniqueness and the Intertwining Number

The definition of the intertwining number $I(\alpha, \beta)$ does not presuppose that G is finite, and so extends to the compact case, as do all the results of this chapter.

3.8 The Character of a Representation

The definition of the character of a finite-dimensional representation does not depend in any way on the finiteness of the group, and so extends to the compact case.

There is one result, however, which extends to this case, but whose proof requires a little more thought.

Proposition 3.1 *Suppose α is an n-dimensional representation of a compact group G over $\mathbb{R}$ or $\mathbb{C}$; and suppose $g \in G$. Let the eigenvalues of $\alpha(g)$ be $\lambda_1, \ldots, \lambda_n$. Then*

$$|\lambda_i| = 1 \qquad (i = 1, \ldots, n).$$

Proof ► We know that there exists an invariant inner product $\langle u, v\rangle$ on the representation-space V. We can choose a basis for V so that

$$\langle v, v\rangle = |x_1|^2 + \cdots + |x_n|^2,$$

where $v = (x_1, \ldots, x_n)'$. Since $\alpha(g)$ leaves this form invariant for each $g \in G$, it follows that the matrix $A(g)$ of $\alpha(g)$ with respect to this basis is orthogonal if $k = \mathbb{R}$, or unitary if $k = \mathbb{C}$.

The result now follows from the fact that the eigenvalues of an orthogonal or unitary matrix all have absolute value 1:

$$\begin{aligned} Uv = \lambda v &\implies v^* U^* = \overline{\lambda} v^* \\ &\implies v^* U^* U v = \overline{\lambda}\lambda v^* v \\ &\implies v^* v = |\lambda|^2 v^* v \\ &\implies |\lambda| = 1. \end{aligned}$$

Hence

$$\lambda^{-1} = \overline{\lambda}$$

for each such eigenvalue. ◄

Alternative proof ► Recall how we proved this in the finite case. By Lagrange's Theorem $g^m = 1$ for some $m > 0$, for each $g \in G$. Hence

$$\alpha(g)^m = I;$$

and so the eigenvalues of $\alpha(g)$ all satisfy

$$\lambda^m = 1.$$

In particular

$$|\lambda| = 1;$$

and so

$$\lambda^{-1} = \bar{\lambda}.$$

We cannot say that an element g in a *compact* group G is necessarily of finite order. However, we *can* show that the powers g^n of g approach arbitrarily close to the identity $e \in G$. (In other words, some subsequence of $\{g, g^2, g^3, \dots\}$ tends to e.)

For suppose not. Then we can find an open set $U \ni e$ such that no power of g except $g^0 = e$ lies in U. Let V be an open neighbourhood of e such that $VV^{-1} \subset U$. Then the subsets $g^n V$ are disjoint. For

$$\begin{aligned} x \in g^m V \cap g^n V &\implies x = g^m v_1 = g^n v_2 \\ &\implies g^{n-m} = v_1 v_2^{-1} \\ &\implies g^{n-m} \in U, \end{aligned}$$

contrary to hypothesis.

It follows [the details are left to the student] that the subgroup

$$\langle g \rangle = \{\dots, g^{-1}, e, g, g^2, \dots\}$$

is

1. discrete,

2. infinite, and

3. closed in G.

But this implies that G has a non-compact closed subgroup, which is impossible.

Thus we can find a subsequence

$$1 \le n_1 < n_2 < \dots$$

such that

$$g^{n_i} \to e$$

as $i \to \infty$.

It follows that

$$\alpha(g)^{n_i} \to I$$

as $i \to \infty$. Hence if λ is any eigenvector of $\alpha(g)$ then

$$\lambda^{n_i} \to 1.$$

This implies in particular that

$$|\lambda| = 1.$$

◄

Corollary 3.1 *If α is a finite-dimensional representation of a compact group over $\mathbb{R}$ or $\mathbb{C}$ then*

$$\chi_\alpha(g^{-1}) = \overline{\chi_\alpha(g)}$$

for all $g \in G$

Proof ► Suppose the eigenvalues of $\alpha(g)$ are $\lambda_1, \ldots, \lambda_n$. Then the eigenvalues of $\alpha(g^{-1}) = \alpha(g)^{-1}$ are $\lambda_1^{-1}, \ldots, \lambda_n^{-1}$. Thus

$$\begin{aligned} \chi_\alpha(g^{-1} &= \operatorname{tr}\alpha(g^{-1}) \\ &= lambda_1^{-1} + \cdots + \lambda_n^{-1} \\ &= \overline{\lambda_1} + \cdots + \overline{\lambda_n} \\ &= \overline{\lambda_1 + \cdots + \lambda_n} \\ &= \overline{\operatorname{tr}\alpha(g)} \\ &= \overline{\chi_\alpha(g)}. \end{aligned}$$

◄

3.9 The Regular Representation

Suppose G is a compact group. We denote by

$$C(G) = C(G, k)$$

(where $k = \mathbb{R}$ or $\mathbb{C}$) the space of all *continuous* maps

$$f : G \to k.$$

If G is discrete (in particular if G is finite) then every map $f : G \to k$ is continuous; so our definition in this case coincides with the earlier one.

If G is not finite then the vector space $C(G, k)$ is infinite-dimensional. [We leave the proof of this to the student.] So if we wish to extend our results from the finite case we are forced to consider *infinite-dimensional* representations. We shall do this, rather briefly, in Chapter 7 below, when we consider the Peter-Weyl Theorem. For the moment, however, we are restricting ourselves to finite-dimensional representations, as we have said; so in this context our results on the regular (and adjoint) representations do *not* extend to the compact case.

As we shall see in Chapter 7, a compact but non-finite group G has an *infinite number* of distinct simple finite-dimensional representations $\sigma_1, \sigma_2, \ldots$. So any argument relying on this number being finite (as for example the proof of the fundamental result on the representations of product-groups, discussed below) cannot be relied on in the compact case.

3.10 Induced Representations

The results of this chapter have only a limited application in the topological case, since they apply only where we have a subgroup $H \subset G$ of *finite index* in G; that is, G is expressible as the union of a finite number of H-cosets:

$$G = g_1 H \cup \cdots \cup g_r H.$$

In this limited case each finite-dimensional representation α of H *induces* a similar representation α^G of G.

For example, $\mathbf{SO}(n)$ is of index 2 in $\mathbf{O}(n)$; so each representation of $\mathbf{SO}(n)$ defines a representation of $\mathbf{O}(n)$.

3.11 Representations of Product Groups

If α, β are finite-dimensional representations of the groups G, H in the vector spaces U, W over k then we have defined the representation $\alpha \times \beta$ of $G \times H$ in $\mathbf{U} \otimes W$. This extends without difficulty to the topological case; and it is a straightforward matter to verify that $\alpha \times \beta$ is *continuous*, in the finite-dimensional case.

Recall our main result in this context; if $k = \mathbb{C}$ then $\alpha \times \beta$ is simple if and only if α and β are both simple; and furthermore, every simple representation of $G \times H$ over $\mathbb{C}$ arises in this way.

The proof that $\alpha \times \beta$ is simple if and only if if and only if α and β are both simple remains valid. However, our first proof that every simple representation of $G \times H$ is of this form fails, although the result is still true.

Let us recall that proof. We argued that if G has m classes, then it has m simple representations $\sigma_1, \ldots, \sigma_m$. Similarly if H has n classes, then it has n simple representations $\tau_1, \ldots, \tau_m$.

But now $G \times H$ has mn classes; and so the mn simple representations $\sigma_i \times \tau_j$ provide all the representations of $G \times H$.

This argument fails in the compact case, since m and n are infinite (unless G or H is finite).

We must turn therefore to our second proof that a simple representation γ of $G \times H$ over $\mathbb{C}$ is necessarily of the form $\alpha \times \beta$. Recall that this alternative proof was based on the natural equivalence

$$\hom(\hom(V, U), W) = \hom(V, U \otimes W).$$

This proof *does* carry over to the compact case.

Suppose the representation-space of γ is the $G \times H$-space V. Consider V as a G-space (ie forget for the moment the action of H on V). Let $U \subset V$ be a simple G-subspace of V. Then there exists a non-zero G-map $t : V \to U$ (since the G-space V is semisimple). Thus the vector space

$$X = \hom^G(V, U)$$

formed by all such G-maps is non-zero.

Now H acts naturally on X:

$$(ht)(v) = t(hv).$$

Thus X is an H-space. Let W be a simple H-subspace of X. Then there exists a non-zero H-map $u : X \to W$ (since the H-space X is semisimple). Thus

$$\hom^H(X, W) = \hom^H\left(\hom^G(V, U), W\right)$$

is non-zero. But it is readily verified that

$$\hom^H\left(\hom^G(V, U), W\right) = \hom^{G \times H}(V, U \otimes W).$$

Thus there exists a non-zero $G \times H$-map $T : V \to U \times W$. Since V and $U \otimes W$ are both simple $G \times H$-spaces, T must be an isomorphism:

$$V = U \otimes W.$$

In particular

$$\gamma = \alpha \times \beta,$$

where α is the representation of G in U, and β is the representation of H in W.

Thus if G and H are compact groups then every simple representation of $G \times H$ over $\mathbb{C}$ is of the form $\alpha \times \beta$.

[Can you see where we have used the fact that G and H are compact in our argument above?]

3.12 Real Representations

Everything in this chapter carries over to the compact case, with no especial problems arising.

Chapter 4

Representations of $\mathrm{U}(1)$

The group $\mathbf{U}(1)$ goes under many names:

$$\mathbf{U}(1) = \mathbf{SO}(2) = S^1 = \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}.$$

Whatever it is called, $\mathbf{U}(1)$ is *abelian*, *connected* and—above all—*compact*.

As an abelian group, every simple representation of $\mathbf{U}(1)$ (over $\mathbb{C}$) is 1-dimensional.

Proposition 4.1 *Suppose $\alpha : G \to \mathbb{C}^\times$ is a 1-dimensional representation of the compact group G. Then*

$$|\alpha(g)| = 1 \textit{ for all } g \in G.$$

Proof ► Since G is compact, so is its continuous image $\alpha(G)$. In particular $\alpha(G)$ is bounded.

Suppose $|\alpha(g)| > 1$. Then

$$|\alpha(g^n)| = |\alpha(g)|^n \to \infty$$

as $n \to \infty$, contradicting the boundedness of $\alpha(G)$.

On the other hand,

$$|\alpha(g)| < 1 \Longrightarrow |\alpha(g^{-1}| = |\alpha(g)|^{-1} > 1.$$

Hence $|\alpha(g)| = 1$. ◄

Corollary 4.1 *Every 1-dimensional representation α of a compact group G is a homomorphism of the form*

$$\alpha : G \to \mathbf{U}(1).$$

In particular, the simple representations of $\mathbf{U}(1)$ are just the homomorphisms

$$\mathbf{U}(1) \to \mathbf{U}(1).$$

But if A is an *abelian* group then for each $n \in \mathbb{Z}$ the map

$$a \mapsto a^n : A \to A$$

is a homomorphism.

Definition 4.1 *For each $n \in \mathbb{Z}$, we denote by E_n the representation*

$$e^{i\theta} \mapsto e^{in\theta}$$

of $\mathbf{U}(1)$.

Proposition 4.2 *The representations E_n are the only simple representations of* $\mathbf{U}(1)$.

Proof ▶ Suppose α is a 1-dimensional representation of $\mathbf{U}(1)$, ie a homomorphism

$$\alpha : \mathbf{U}(1) \to \mathbf{U}(1).$$

Let $U \subset \mathbf{U}(1)$ be the open set

$$U = \{e^{i\theta} : -\pi/2 < \theta < \pi/2\}.$$

Note that each $g \in U$ has *a unique square root* in U, ie there is one and only one $h \in U$ such that $h^2 = g$.

Since α is continuous at 1, we can find $\delta > 0$ such that

$$-\delta < \theta < \delta \Longrightarrow \alpha(e^{i\theta}) \in U.$$

Choose N so large that $1/N < \delta$. Let $\omega = e^{2\pi i/N}$. Then $\alpha(\omega) \in U$; while

$$\omega^N = 1 \Longrightarrow \alpha(\omega)^N = 1.$$

It follows that

$$\alpha(\omega) = e^{2\pi ni/N} = \omega^n = E_n(\omega)$$

for some $n \in \mathbb{Z}$ in the range $-N/2 < n < N/2$. We shall deduce from this that $\alpha = E_n$.

Let

$$\omega_1 = e^{\pi i/N}.$$

Then

$$\begin{aligned} \omega_1^2 = \omega \quad &\Longrightarrow \quad \alpha(\omega_1)^2 = \alpha(\omega) = \omega^n \\ &\Longrightarrow \quad \alpha(\omega_1) = \omega_1^n, \end{aligned}$$

since this is the unique square root of ω^n in U.

Repeating this argument successively with we deduce that if

$$\omega_j = e^{\frac{2\pi}{2^j N} i}$$

then

$$\alpha(\omega_j) = \omega_j^n = E_n(\omega_j)$$

for $j = 2, 3, 4, \ldots$.

But it follows from this that

$$\alpha(\omega_j^k) = (\omega_j^k)^n = E_n(\omega_j^k)$$

for $k = 1, 2, 3, \ldots$. In other words

$$\alpha(e^{i\theta}) = E(e^{i\theta})$$

for all θ of the form

$$\theta = 2\pi \frac{k}{2^j}$$

But these elements $e^{i\theta}$ are dense in $\mathbf{U}(1)$. Therefore, by continuity,

$$\alpha(g) = E_n(g)$$

for all $g \in \mathbf{U}(1)$, ie $\alpha = E_n$. ◄

Alternative proof ► Suppose

$$\alpha : \mathbf{U}(1) \to \mathbf{U}(1)$$

is a representation of $\mathbf{U}(1)$ distinct from all the E_n. Then

$$I(E_n, \alpha) = 0$$

for all n, ie

$$c_n = \frac{1}{2\pi} \int_0^{2\pi} \alpha(e^{i\theta}) e^{-n\theta} \, d\theta = 0.$$

In other words, *all the Fourier coefficients of* $\alpha(e^{i\theta})$ *vanish.*

But this implies (from Fourier theory) that the function itself must vanish, which is impossible since $\alpha(1) = 1$. ◄

Remark: As this proof suggests, the representation theory of $\mathbf{U}(1)$ is just the Fourier theory of periodic functions in disguise. (In fact, the whole of group representation theory might be described as a kind of generalised Fourier analysis.)

Let ρ denote the representation of $\mathbf{U}(1)$ in the space $C(\mathbf{U}(1))$ of continuous functions $f : \mathbf{U}(1) \to \mathbb{C}$, with the usual action: if $g = e^{i\phi}$ then

$$(gf)(e^{i\theta}) = e^{i(\theta-\phi)}.$$

The Fourier series

$$f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} c_n e^{in\theta}$$

expresses the splitting of $C(\mathbf{U}(1))$ into 1-dimensional spaces

$$C\left(\mathbf{U}(1)\right) = \bigoplus V_n,$$

where

$$V_n = \langle e^{in\theta} \rangle = \{ce^{in\theta} : c \in \mathbb{C}\}.$$

Notice that with our definition of group action, *the space V_n carries the representation E_{-n}*, rather than E_n. For if $g = e^{i\phi}$, and $f(e^{i\theta}) = e^{in\theta}$, then

$$(gf)(e^{i\theta}) = e^{-in\phi} f(e^{i\theta}) = E_{-n}(g) f(e^{i\theta}).$$

In terms of representations, the splitting of $C(\mathbf{U}(1))$ may be written:

$$\rho = \sum_{n\in\mathbb{Z}} E_n.$$

We must confess at this point that we have gone ‘out of bounds’ in these remarks, since the vector space $C(G)$ is *infinite-dimensional* (unless G is finite), whereas all our results to date have been restricted to finite-dimensional representations. We shall see in Chapter 7 how we can justify this extension.

Chapter 5

Representations of $\mathbf{SU}(2)$

5.1 Conjugacy in $\mathbf{SU}(n)$

Since characters are class functions, our first step in studying the representations of a compact group G—as of a finite group—is to determine how G divides into conjugacy classes.

We know that if 2 matrices $S, T \in \mathbf{GL}(n, k)$ are *similar*, ie conjugate in $\mathbf{GL}(n, k)$, then they will have the same eigenvalues $\lambda_1, \ldots, \lambda_n$. So this gives a *necessary* condition for conjugacy in any matrix group $G \subset \mathbf{GL}(n, k)$:

$$S \sim T \text{ (in } G) \Longrightarrow S, T \text{ have same eigenvalues.}$$

In general this condition is not sufficient, eg

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \not\sim \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

in $\mathbf{GL}(2, C)$, although both matrices have eigenvalues $1, 1$. However we shall see that the condition *is* sufficient in each of the classical compact matrix groups $\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{Sp}(n)$.

Two remarks: Firstly, when speaking of conjugacy we must always be clear *in what group we are taking conjugates*. Two matrices $S, T \in G \subset \mathbf{GL}(n, k)$ may well be conjugate in $\mathbf{GL}(n, k)$ without being conjugate in G.

Secondly, the concepts of eigenvalue and eigenvector really belong to a *representation* of a group rather than the group itself. So for example, when we speak of an eigenvalue of $T \in \mathbf{U}(n)$ we really should—though we rarely shall—say *an eigenvalue of T in the natural representation of $\mathbf{U}(n)$ in $\mathbb{C}^n$*.

Lemma 5.1 *The diagonal matrices in $\mathbf{U}(n)$ form a subgroup isomorphic to the torus group $\mathbb{T}^n \equiv \mathbf{U}(1)^n$.*

Proof ► We know that the eigenvalues of $T \in \mathbf{U}(n)$ have absolute value 1, since

$$\begin{aligned} Tv = \lambda v &\implies v^*T = \bar{\lambda}v^* \\ &\implies v^*T^*Tv = \bar{\lambda}\lambda v^*v \\ &\implies v^*v = \bar{\lambda}\lambda v^*v \\ &\implies |\lambda|^2 = \bar{\lambda}\lambda = 1 \\ &\implies |\lambda| = 1 \end{aligned}$$

Thus the eigenvalues of T can be written in the form

$$e^{i\theta_1}, \dots, e^{i\theta_n} \quad (\theta_1, \dots, \theta_n \in \mathbb{R}).$$

In particular the diagonal matrices in $\mathbf{U}(n)$ are just the matrices

$$\begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

It follows that the homomorphism

$$\mathbf{U}(1)^n \to \mathbf{U}(n) : \left(e^{\theta_1}, \dots, e^{\theta_n}\right) \mapsto \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

maps $\mathbf{U}(1)^n$ homeomorphically onto the diagonal subgroup of $\mathbf{U}(n)$, allowing us to identify the two:

$$\mathbf{U}(1)^n \subset \mathbf{U}(n).$$

◄

Lemma 5.2 *Every unitary matrix* $T \in \mathbf{U}(n)$ *is conjugate (in* $\mathbf{U}(n)$*) to a diagonal matrix:*

$$T \sim D \in \mathbf{U}(1)^n.$$

Remark: You are probably familiar with this result: *Every unitary matrix can be diagonalised by a unitary transformation.* But it is instructive to give a proof in the spirit of representation theory.

Proof ► Let $\langle T \rangle$ denote the closed subgroup generated by T, ie the closure in $U(n)$ of the group

$$\{\dots, T^{-1}, I, T, T^2, \dots\}$$

formed by the powers of T.

This group is abelian; and its natural representation in $\mathbb{C}^n$ leaves invariant the standard positive-definite hermitian form $|x_1|^2 + \cdots + |x_n|^2$, since it consists of unitary matrices.

It follows that this representation splits into a sum of 1-dimensional representations, mutually orthogonal with respect to the standard form. If we choose a vector e_i of norm 1 in each of these 1-dimensional spaces we obtain an orthonormal set of eigenvectors of T. If U is the matrix of change of basis, ie

$$U = (e_1, \ldots, e_n)$$

then

$$U^*TU = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

where

$$Te_i = e^{\theta_i} e_i.$$

◀

Lemma 5.3 *The diagonal matrices in* $\mathbf{SU}(n)$ *form a subgroup isomorphic to the torus group* $\mathbb{T}^{n-1} \equiv \mathbf{U}(1)^{n-1}$.

Proof ▶ If

$$T = \begin{pmatrix} e^{i\theta_1} & & \\ & \ddots & \\ & & e^{i\theta_n} \end{pmatrix}$$

then

$$\det T = e^{i(\theta_1 + \cdots + \theta_n)}.$$

Hence

$$T \in \mathbf{SU}(n) \iff \theta_1 + \cdots + \theta_n = 0 \pmod{2\pi}.$$

Thus the homomorphism

$$\mathbf{U}(1)^{n-1} \to \mathbf{SU}(n) : \left(e^{\theta_1}, \ldots, e^{\theta_{n-1}}\right) \mapsto \begin{pmatrix} e^{i\theta_1} & & & \\ & \ddots & & \\ & & e^{i\theta_{n-1}} & \\ & & & e^{-i(\theta_1 + \cdots + \theta_{n-1})} \end{pmatrix}$$

maps $\mathbf{U}(1)^{n-1}$ homeomorphically onto the diagonal subgroup of $\mathbf{SU}(n)$, allowing us to identify the two:

$$\mathbf{U}(1)^{n-1} \subset \mathbf{SU}(n).$$

◀

Lemma 5.4 *Every matrix* $T \in \mathbf{SU}(n)$ *is conjugate (in* $\mathbf{SU}(n)$*) to a diagonal matrix:*

$$T \sim D \in \mathbf{U}(1)^{n-1}.$$

Proof ► From the corresponding lemma for $\mathbf{U}(n)$ above, T is conjugate *in the full group* $\mathbf{U}(n)$ to a diagonal matrix:

$$U^*TU = D \qquad (U \in \mathbf{U}(n)).$$

We know that $|\det U| = 1$, say

$$\det U = e^{i\phi}.$$

Let

$$V = e^{-i\phi/n}U.$$

Then $V \in \mathbf{SU}(n)$; and

$$V^*TV = D.$$

◄

Lemma 5.5 *Let* $G = \mathbf{U}(n)$ *or* $\mathbf{SU}(n)$*. Two matrices* $U, V \in G$ *are conjugate if and only if they have the same eigenvalues*

$$\{e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_n}\}.$$

Proof ► Suppose $U, V \in G$. If $U \sim V$ then certainly they must have the same eigenvalues.

Conversely, suppose $U, V \in G$ have the same eigenvalues. As we have seen, U and V are each conjugate in G to diagonal matrices:

$$U \sim D_1,\ V \sim D_2.$$

The entries in the diagonal matrices are just the eigenvalues. Thus D_1 and D_2 contain the same entries, perhaps permuted. So we can find a permutation matrix P (with just one 1 in each row and column, and 0's elsewhere) such that

$$D_2 = P^{-1}D_1P.$$

Now $P \in \mathbf{U}(n)$ since permutation of coordinates clearly leaves the form $|x_1|^2 + \cdots + |x_n|^2$ unchanged. Thus if $G = \mathbf{U}(n)$ we are done:

$$S \sim D_1 \sim D_2 \sim T.$$

Finally, suppose $G = \mathbf{SU}(n)$. Then

$$T = U^* S U$$

for some $U \in \mathbf{U}(n)$. Suppose

$$\det U = e^{i\phi}.$$

Let

$$V = e^{-i\phi/n} U.$$

Then $V \in \mathbf{SU}(n)$; and

$$T = V^* S V$$

◀

5.2 Representations of $\mathbf{SU}(2)$

Summarising the results above, as they apply to $\mathbf{SU}(2)$:

1. each $T \in \mathbf{SU}(2)$ has eigenvalues $e^{\pm i\theta}$
2. with the same notation,

$$T \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

3. $U(-\theta) \sim U(\theta)$

Thus $\mathbf{SU}(2)$ divides into classes $C(\theta)$ (for $0 \le \theta \le \pi$) containing all T with eigenvalues $e^{\pm i\theta}$.

The classes

$$C(0) = \{I\}, \quad C(1) = \{-I\},$$

constituting the centre of $SU(2)$, each contain a single element; all other classes are infinite, and intersect the diagonal subgroup in 2 elements:

$$C(\theta) \cap \mathbf{U}(1) = \{U(\pm\theta)\}.$$

Now let ρ denote the natural representation of $\mathbf{SU}(2)$ in $\mathbb{C}^2$, defined by the action

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = T \begin{pmatrix} z \\ w \end{pmatrix}$$

Explicitly, recall that the matrices $T \in \mathbf{SU}(2)$ are just those of the form

$$U = \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \qquad (|a|^2 + |b|^2 = 1)$$

Taking T in this form, its action is given by

$$(z, w) \mapsto (az + bw, -\bar{b}z + \bar{a}w)$$

By extension, this change of variable defines an action of $\mathbf{SU}(2)$ on polynomials $P(z, w)$ in z and w:

$$P(z, w) \mapsto P(az + bw, -\bar{b}z + \bar{a}w).$$

Definition 5.1 *For each half-integer* $j = 0, 1/2, 1/, 3/2, \ldots$ *we denote by* D_j *the representation of* $\mathbf{SU}(2)$ *in the space*

$$V(j) = \langle z^{2j}, z^{2j-1}w, \ldots, w^{2j} \rangle$$

of homogeneous polynomials in z, w *of degree* $2j$*.*

Example: Let $j = 3/2$. The 4 polynomials

$$z^3, z^2w, zw^2, w^3$$

form a basis for V(3/2).

Consider the action of the matrix

$$T = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in \mathbf{SU}(2).$$

We have

$$\begin{aligned} T(z^3) &= (iw)^3 = -iw^3, \\ T(z^2w) &= -izw^2, \\ T(zw^2) &= -iz^2w, \\ T(w^3) &= -iz^3 \end{aligned}$$

Thus under $D_{\frac{3}{2}}$,

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

Proposition 5.1 *The character χ_j of D_j is given by the following rule: Suppose T has eigenvalues $e^{\pm i\theta}$ Then*

$$\chi_j(T) = e^{2ij\theta} + e^{2i(j-1)\theta} + \cdots + e^{-2ij\theta}$$

Proof ► We know that

$$T \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

Hence

$$\chi_j(T) = \chi_j\left(U(\theta)\right).$$

The result follows on considering the action of $U(\theta)$ on the basis $\{z^{2j}, \ldots, w^{2j}\}$ of $V(j)$. For

$$\begin{aligned} U(\theta) z^k w^{2j-k} &= \left(e^{i\theta} z\right)^k \left(e^{-i\theta} w\right)^{2j-k} \\ &= e^{2i(k-j)\theta} z^k w^{2j-k}. \end{aligned}$$

Thus under D_j,

$$U(\theta) \mapsto \begin{pmatrix} e^{2ij\theta} & & & \\ & e^{2i(j-2)\theta} & & \\ & & \ddots & \\ & & & e^{-2ij\theta} \end{pmatrix}$$

whence

$$\chi_j\left(U(\theta)\right) = e^{2ij\theta} + e^{2i(j-2)\theta} + \cdots + e^{-2ij\theta}.$$

◄

Proposition 5.2 *For each half-integer j, D_j is a simple representation of* $\mathbf{SU}(2)$, *of dimension* $2j+1$.

Proof ► On restricting to the diagonal subgroup $\mathbf{U}(1) \subset \mathbf{SU}(2)$,

$$(D_j)_{\mathbf{U}(1)} = E_{-2j} + E_{-2j+2} + \cdots + E_{2j}.$$

Since the simple parts on the right are distinct, it follows that the corresponding expression

$$V(j) = \langle z^{2j} \rangle \oplus \cdots \oplus \langle w^{2j} \rangle$$

for $V(j)$ as a direct sum of simple $\mathbf{U}(1)$-modules is unique.

Now suppose that $V(j)$ splits as an $\mathbf{SU}(2)$-module, say

$$V(j) = U \oplus W.$$

If we expressed U and W as direct sums of simple $\mathbf{U}(1)$-spaces, we would obtain an expression for $V(j)$ as a direct sum of simple $\mathbf{U}(1)$-spaces. It follows from the uniqueness of this expression that each of U and W must be the spaces spanned by some of the monomials $z^a w^b$. In particular z^{2j} must belong either to U or to W. Without loss of generality we may suppose that

$$z^{2j} \in U$$

But then

$$T(z^{2j}) \in U$$

for all $T \in \mathbf{SU}(2)$. In particular, taking

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(almost any T would do) we see that

$$(z+w)^{2j} = z^{2j} + 2jz^{2j-1}w + \cdots + w^{2j} \in U.$$

Each of the monomials of degree $2j$ occurs here with non-zero coefficient. It follows that each of these monomials must be in U:

$$z^{2j-k}w^k \in U \quad \text{for all } k.$$

Hence $U = V(j)$, ie D_j is simple. ◄

Proposition 5.3 *The D_j are the only simple representations of* $\mathbf{SU}(2)$.

Proof ► Suppose α is a simple representation of $\mathbf{SU}(2)$ distinct from the D_j. Then in particular

$$I(\alpha, D_j) = 0.$$

In other words, χ_α is orthogonal to each χ_j,

Consider the restriction of α to the diagonal subgroup $\mathbf{U}(1)$. Suppose

$$\alpha_{\mathbf{U}(1)} = \sum_j n_j E_j,$$

where of course all but a finite number of the n_j vanish (and the rest are positive integers). It follows that

$$\chi_\alpha(U(\theta)) = \sum_j n_j e^{ij\theta}$$

Lemma 5.6 *For any representation* α *of* $\mathbf{SU}(2)$*,*

$$n_{-j} = n_j,$$

ie E_j *and* E_{-j} *occur with the same multiplicity in* $\alpha_{\mathbf{U}(1)}$.

Proof ► This follows at once from the fact that

$$U(-\theta) \sim U(\theta)$$

in $\mathbf{SU}(2)$. ◄

Since $n_{-j} = n_j$, we see that $\chi_\alpha(U(\theta))$ is expressible as a linear combination of the $\chi_j(U(\theta))$ (in fact with integral—and not necessarily positive—coefficients):

$$\chi_\alpha\left(U(\theta)\right) = \sum_j c_j \chi_j\left(U(\theta)\right).$$

Since each $T \in \mathbf{SU}(2)$ is conjugate to some $U(\theta)$ it follows that

$$\chi_\alpha(T) = \sum_j c_j \chi_j(T)$$

for all $T \in \mathbf{SU}(2)$. But this contradicts the proposition that the simple characters are linearly independent (since they are orthogonal). ◄

We know that every finite-dimensional representation of $\mathbf{SU}(2)$ is semi-simple. In particular, each product $D_j D_k$ is expressible as a sum of simple representations, ie as a sum of D_n's.

Theorem 5.1 (The Clebsch-Gordan formula) *For any pair of half-integers* j, k

$$D_j D_k = D_{j+k} + D_{j+k-1} + \cdots + D_{|j-k|}.$$

Proof ► We may suppose that $j \geq k$.

Suppose T has eigenvalues $e^{\pm i\theta}$. For any 2 half-integers a, b such that $a \leq b,\ a - b \in \mathbb{N}$, let

$$L(a, b) = e^{2ia\theta} + e^{2i(a+1)\theta} + \cdots + e^{2ib\theta}.$$

(We may think of $L(a, b)$ as a 'ladder' linking a to b on the axis, with 'rungs' every step, at $a + 1, a + 2, \ldots$.) Thus

$$\chi_j(\theta) = L(-j, j);$$

and so

$$\chi_{D_j D_k}(T) = \chi_j(\theta)\chi_k(\theta) = L(-j, j)L(-k, k).$$

We have to show that

$$L(-j,j)L(-k,k) = L(-j-k,j+k)+L(-j-k+1,j+k-1)+\cdots+L(-j+k,j-k).$$

We argue by induction on k. The result holds trivially for $k=0$.

By our inductive hypothesis,

$$L(-j,j)L(,-k+1,k-1) = L(-j-k+1,j+k-1)+\cdots+L(-j+k-1,j-k+1).$$

Now

$$L(k) = L(k-1) + (e^{-2ik\theta} + e^{2ik\theta}).$$

But

$$\begin{aligned} L(-j,j)e^{-2ik\theta} &= L(-j-k,j-k), \\ L(-j,j)e^{2ik\theta} &= L(-j+k,j+k). \end{aligned}$$

Thus

$$\begin{aligned} L(-j,j)(e^{-2ik\theta}+e^{2ik\theta}) &= L(-j-k,j-k)+L(-j+k,j+k) \\ &= L(-j-k,j+k)+L(-j+k,j-k). \end{aligned}$$

Gathering our ladders together,

$$\begin{aligned} L(-j,j)L(-k,k) &= L(-j-k+1,j+k-1)+\cdots+L(-j+k-1,j-k+1) \\ &\quad +L(-j-k,j+k)+L(-j+k,j-k) \\ &= L(-j-k,j+k)+\cdots+L(-j+k,j-k), \end{aligned}$$

as required. ◄

Proposition 5.4 *The representation D_j of $\mathbf{SU}(2)$ is real for integral j and quaternionic for half-integral j.*

Proof ► The character

$$\chi_j(\theta) = e^{2ij\theta} + e^{2i(j-1)\theta} + \cdots + e^{-2ij\theta}$$

is real, since

$$\overline{\chi_j(\theta)} = e^{-2ij\theta} + e^{-2i(j-1)\theta} + \cdots + e^{2ij\theta} = \chi_j(\theta).$$

Thus D_j (which we know to be simple) is either real or quaternionic.

A quaternionic representation always has even dimension; for it carries an invariant non-singular skew-symmetric form, and such a form can only exist in even dimension, since it can be reduced to the form

$$x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3 + \cdots$$

But

$$\dim D_j = 2j + 1$$

is odd for integral j. Hence D_j must be real in this case.

Lemma 5.7 *The representation $D_{\frac{1}{2}}$ is quaternionic.*

Proof of Lemma ▷ Suppose $D_{\frac{1}{2}}$ were real, say

$$D_{\frac{1}{2}} = \mathbb{C}\beta,$$

where

$$\beta : \mathbf{SU}(2) \to \mathbf{GL}(2, \mathbb{R})$$

is a 2-dimensional representation of $\mathbf{SU}(2)$ over $\mathbb{R}$. We know that this representation carries an invariant positive-definite form. By change of coordinates we can bring this to $x_1^2 + x_2^2$, so that

$$\operatorname{im}\beta \subset \mathbf{O}(2).$$

Moreover, since $\mathbf{SU}(2)$ is connected, so is its image. Hence

$$\operatorname{im}\beta \subset \mathbf{SO}(2).$$

Thus β defines a homomorphism

$$\mathbf{SU}(2) \to \mathbf{SO}(2) = \mathbf{U}(1),$$

ie a 1-dimensional representation γ of $\mathbf{SU}(2)$, which must in fact be $D_0 = 1$. It follows that $\beta = 1 + 1$, contradicting the simplicity of $D_{\frac{1}{2}}$. ◁

Remark: It is worth noting that the representation $D_{\frac{1}{2}}$ is quaternionic in its original sense, in that it arises from a representation in a quaternionic vector space. To see this, recall that

$$\mathbf{SU}(2) = \mathbf{Sp}(1) = \{q \in \mathbb{H} : |q| = 1\}.$$

The symplectic group $\mathbf{Sp}(1)$ acts naturally on $\mathbb{H}$, by left multiplication:

$$(g, q) \mapsto gq \quad (g \in \mathbf{Sp}(1),\ q \in \mathbb{H}).$$

(We take scalar multiplication in quaternionic vector spaces on the right.) It is easy to see that this 1-dimensional representation over $\mathbb{H}$ gives rise, on restriction of scalars, to a simple 2-dimensional representation over $\mathbb{C}$, which must be $D_{\frac{1}{2}}$.

It remains to prove that D_j is quaternionic for half-integral $j > \frac{1}{2}$. Suppose in fact D_j were real; and suppose this were the first half-integral j with that property. Then

$$D_j D_1 = D_{j+1} + D_j + D_{j-1}$$

would also be real (since the product of 2 real representations is real). But D_{j-1} is quaternionic, by assumption, and so must appear with even multiplicity in any real representation. This is a contradiction; so D_j must be quaternionic for all half-integral j. ◀

Alternative Proof ▶ Recall that if α is a simple representation then

$$\int \chi_\alpha(g^2)\, dg = \begin{cases} 1 & \text{if } \alpha \text{ is real,} \\ 0 & \text{if } \alpha \text{ is essentially complex,} \\ -1 & \text{if } \alpha \text{ is quaternionic.} \end{cases}$$

Let $\alpha = D_j$. Suppose $g \in \mathbf{SU}(2)$ has eigenvalues $e^{\pm i\theta}$. Then g^2 has eigenvalues $e^{\pm 2i\theta}$, and so

$$\begin{aligned} \chi_j(g^2) &= e^{4ij\theta} + e^{4i(j-1)\theta} + \cdots + e^{-4ij\theta} \\ &= \chi_{2j}(g) - \chi_{2j-1}(g) + \cdots + (-1)^{2j}\chi_0(g). \end{aligned}$$

Thus

$$\begin{aligned} \int \chi_j(g^2)\, dg &= \int \chi_{2j}(g)\, dg - \int \chi_{2j-1}(g)\, dg + \cdots + (-1)^{2j} \int \chi_0(g)\, dg \\ &= I(1, D_{2j}) - I(1, D_{2j-1}) + \cdots + (-1)^{2j} I(1, D_0) \\ &= (-1)^{2j} I(1,1) \\ &= \begin{cases} +1 & \text{if } j \text{ is integral} \\ -1 & \text{if } j \text{ is half-integral} \end{cases} \end{aligned}$$

◀

Chapter 6

Representations of $\mathbf{SO}(3)$

Definition 6.1 *A* covering *of one topological group G by another C is a continuous homomorphism*

$$\Theta : C \to G$$

such that

1. $\ker \Theta$ *is discrete;*
2. Θ *is surjective, ie* $\operatorname{im}\Theta = G$.

Proposition 6.1 *A discrete subgroup is necessarily closed.*

Proof ► Suppose $S \subset G$ is a discrete subgroup. Then by definition we can find an open subset $U \subset G$ such that

$$U \cap S = \{1\}.$$

(For if S is discrete then $\{1\}$ is open in the induced topology on S, ie it is the intersection of an open set in G with S.)

We can find an open set $V \subset G$ containing 1 such that

$$V^{-1}V \subset U,$$

ie $v_1^{-1}v_2 \in U$ for all $v_1, v_2 \in V$. This follows from the continuity of the map

$$(x, y) \mapsto x^{-1}y : G \times G \to G.$$

Now suppose $g \in G \setminus S$. We must show that there is an open set $O \ni g$ not intersecting S. The open set $gV \ni g$ contains at most 1 element of S. For suppose $s, t \in gV$, say

$$s = gv_1,\ t = gv_2.$$

Then

$$s^{-1}t = v_1^{-1}v_2 \in U \cap S.$$

Thus $s^{-1}t = 1$, ie $s = t$.

If $gV \cap S = \emptyset$ then we can take $O = gV$. Otherwise, suppose $gV \cap S = \{s\}$. We can find an open set $W \subset G$ such that $g \in W, s \notin W$; and then we can take $O = gV \cap W$. ◄

Corollary 6.1 *A discrete subgroup of a compact group is necessarily finite.*

Remark: We say that

$$\Theta : C \to G$$

is an n-*fold* covering if $\|\ker\Theta\| = n$.

Proposition 6.2 *Suppose $\Theta : C \to G$ is a surjective (and continuous) homomorphism of topological groups. Then*

1. *Each representation α of G in V defines a representation α' of C in V by the composition*
$$\alpha' : C \xrightarrow{\Theta} G \xrightarrow{\alpha} \mathbf{GL}(V).$$
2. *If the representations α_1, α_2 of G define the representations α_1', α_2' of C in this way then*
$$\alpha_1' = \alpha_2' \iff \alpha_1 = \alpha_2.$$
3. *With the same notation,*
$$(\alpha_1 + \alpha_2)' = \alpha_1' + \alpha_2', \ (\alpha_1\alpha_2)' = \alpha_1'\alpha_2', \ (\alpha^*)' = (\alpha')^*.$$
4. *A representation β of C arises in this way from a representation of G if and only if it is trivial on* $\ker\Theta$*, ie*
$$g \in \ker\Theta \implies \beta(g) = 1.$$
5. *The representation α' of C is simple if and only if α is simple. Moreover, if that is so then α' is real, quaternionic or essentially complex if and only if the same is true of α.*
6. *The representation α' of C is semisimple if and only if α is semisimple.*

Proof ► All follows from the fact that gv $(g \in G,\ v \in V)$ is the same whether defined through α or α'. ◄

Remark: We can express this succinctly by saying that *the representation-ring of G is a sub-ring of the representation-ring of C*:

$$R(G,k) \subset R(C,k).$$

We can identify a representation α of G with the corresponding representation α' of C; so that the representation theory of G is included, in this sense, in the representation theory of C.

The following result allows us, by applying these ideas, to determine the representations of $\mathbf{SO}(3)$ from those of $SU(2)$.

Proposition 6.3 *There exists a two-fold covering*

$$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3).$$

Remark: We know that $\mathbf{SU}(2)$ has the real 2-dimensional representation D_1, defined by a homomorphism

$$\Theta : \mathbf{SU}(2) \to \mathbf{GL}(3,\mathbb{R}).$$

Since $\mathbf{SU}(2)$ is compact, the representation-space carries an invariant positive-definite quadratic form. Taking this in the form $x^2+y^2+z^2$, we see that

$$\operatorname{im}\Theta \subset \mathbf{O}(3).$$

Moreover, since $\mathbf{SU}(2)$ is connected, so is its image. Thus

$$\operatorname{im}\Theta \subset \mathbf{SO}(3).$$

This is indeed the covering we seek; but we prefer to give a more constructive definition.

Proof ► Let $\mathcal{H}$ denote the space of all 2×2 hermitian matrices, ie all matrices of the form

$$A = \begin{pmatrix} x & y-iz \\ y+iz & t \end{pmatrix} \qquad (x,y,z,t \in \mathbb{R}).$$

Evidently $\mathcal{H}$ is a 4-dimensional real vector space. (It is not a complex vector space, since A hermitian does not imply that iA is hermitian; in fact

$$A \text{ hermitian} \implies iA \text{ skew-hermitian},$$

since $(iA)^* = -iA^*$ for any A.)

Now suppose $U \in \mathbf{SU}(2)$. Then

$$\begin{aligned} A \in \mathcal{H} &\implies (U^*AU)^* = U^*A^*U^{**} = U^*AU \\ &\implies U^*AU \in \mathcal{H}. \end{aligned}$$

Thus the action

$$(U, A) \mapsto U^*AU = U^{-1}AU$$

of $\mathbf{SU}(2)$ on $\mathcal{H}$ defines a 4-dimensional real representation of $\mathbf{SU}(2)$.

This is not quite what we want; we are looking for a 3-dimensional representation. Let

$$\mathcal{T} = \{\mathcal{A} \in \mathcal{H} : \operatorname{tr} \mathcal{A} = 0\}$$

denote the subspace of $\mathcal{H}$ formed by the *trace-free* hermitian matrices, ie those of the form

$$A = \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix} \qquad (x, y, z, t \in \mathbb{R}).$$

These constitute a 3-dimensional real vector space; and since

$$\operatorname{tr}(U^*AU) = \operatorname{tr}\left(U^{-1}AU\right) = \operatorname{tr} A$$

this space is stable under the action of $\mathbf{SU}(2)$. Thus we have constructed a 3-dimensional representation of $\mathbf{SU}(2)$ over $\mathbb{R}$, defined by a homomorphism

$$\Theta : \mathbf{SU}(2) \to \mathbf{GL}(3, \mathbb{R}).$$

The determinant defines a negative-definite quadratic form on $\mathcal{T}$, since

$$\det \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix} = -x^2 - y^2 - z^2$$

Moreover this quadratic form is left invariant by the action of $\mathbf{SU}(2)$, since

$$\det(U^*AU) = \det\left(U^{-1}AU\right) = \det A.$$

In other words, $\mathbf{SU}(2)$ acts by orthogonal transformations on $\mathcal{T}$, so that

$$\operatorname{im}\Theta \subset \mathbf{O}(3).$$

Moreover, since $\mathbf{SU}(2)$ is connected, its image must also be connected, and so

$$\operatorname{im}\Theta \subset \mathbf{SO}(3).$$

We use the same symbol to denote the resulting homomorphism

$$\Theta : \mathbf{SU}(2) \to \mathbf{SO}(3).$$

We have to show that this homomorphism is a covering.

Lemma 6.1 $\ker\Theta = \{\pm I\}$.

Proof of Lemma ▷ Suppose $\mathbf{U} \in \ker\Theta$. In other words,

$$U^{-1}AU = A$$

for all $A \in \mathcal{T}$.

In fact this will hold for all hermitian matrices $A \in \mathcal{H}$ since

$$\mathcal{H} = \mathcal{T} \bigoplus \langle \mathcal{I} \rangle.$$

But now the result holds also for all skew-hermitian matrices, since they are of the form iA, with A hermitian. Finally the result holds for *all* matrices $A \in \mathbf{GL}(2, \mathbb{C})$, since every matrix is a sum of hermitian and skew-hermitian parts:

$$A = \frac{1}{2}\left(A + A^*\right) + \frac{1}{2}\left(A - A^*\right).$$

Since

$$U^{-1}AU = A \iff AU = UA,$$

we are looking for matrices U which commute with all 2×2-matrices A. It is readily verified that the only such matrices are the scalar multiples of the identity, ie

$$U = \rho I.$$

But now,

$$\begin{aligned} U \in \mathbf{SU}(2) &\implies \det U = 1 \\ &\implies \rho^2 = 1 \\ &\implies \rho = \pm 1. \end{aligned}$$

◁

Lemma 6.2 *The homomorphism Θ is surjective.*

Proof of Lemma ▷ Let us begin by looking at a couple of examples. Suppose first

$$U = U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix};$$

and suppose

$$A = \begin{pmatrix} x & y - iz \\ y + iz & -x \end{pmatrix} \in \mathcal{T}.$$

Then

$$\begin{aligned}
U^*AU &= \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} x & y-iz \\ y+iz & -x \end{pmatrix} \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \\
&= \begin{pmatrix} x & e^{-2i\theta}(y-iz) \\ e^{2i\theta}(y+iz) & -x \end{pmatrix} \\
&= \begin{pmatrix} X & Y-iZ \\ Y+iZ & -X \end{pmatrix},
\end{aligned}$$

where

$$\begin{aligned}
X &= x \\
Y &= \cos 2\theta y + \sin 2\theta z \\
Z &= \sin 2\theta y - \cos 2\theta z.
\end{aligned}$$

Thus $U(\theta)$ induces a rotation in the space $\mathcal{T}$ through 2θ about the Ox-axis, say

$$U(\theta) \mapsto R(2\theta, Ox).$$

As another example, let

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix};$$

In this case

$$V^*AV = \begin{pmatrix} x & y+iz \\ y-iz & -x \end{pmatrix} = \begin{pmatrix} X & Y+iZ \\ Y-iZ & -X \end{pmatrix},$$

where

$$\begin{aligned}
X &= -y \\
Y &= x \\
Z &= z.
\end{aligned}$$

Thus $\Theta(V)$ is a rotation through $\pi/2$ about Oz.

It is sufficient now to show that the rotations $R(\phi, Ox)$ about the x-axis, together with $T = R(\pi/2, Oz)$, generate the group $\mathbf{SO}(3)$. Since Θ is a homomorphism, $VU(\theta)V^{-1}$ maps onto

$$TR(2\theta, Ox)T^{-1} = R(2\theta, T(Ox)) = R(2\theta, Oy).$$

Thus $\operatorname{im}\Theta$ contains all rotations about Ox and about Oy. It is easy to see that these generate all rotations. For consider the rotation $R(\phi, l)$ about the axis l. We can find a rotation S about Ox bringing the axis l into the plane Oxz; and then a rotation T about Oy bringing l into the coordinate axis Ox. Thus

$$TSR(\phi,l)(TS)^{-1} = R(\phi, Ox);$$

and so

$$R(\phi,l) = S^{-1}T^{-1}R(\phi,Ox)TS.$$

◁

These 2 lemmas show that Θ defines a covering of $\mathbf{SO}(3)$ by $\mathbf{SU}(2)$. ◀

Remarks:

1. We may express this result in the succinct form:

$$\mathbf{SO}(3) = \mathbf{SO}(2)/\{\pm I\}.$$

   Recall that $\mathbf{SU}(2) \cong S^3$. The result shows that $\mathbf{SO}(3)$ is homeomorphic to the space resulting from identifying antipodal points on the sphere S^3. Another way of putting this is to say that $\mathbf{SO}(3)$ is homeomorphic to 3-dimensional real projective space:

$$\mathbf{SO}(3) \cong P^3(\mathbb{R}) = (\mathbb{R}^4 \setminus \{0\})/\mathbb{R}^\times.$$

2. We shall see in Part 4 that the space $\mathcal{T}$ (or more accurately the space $i\mathcal{T}$) is just *the Lie algebra* of the group $\mathbf{SU}(2)$. Every Lie group acts on its own Lie algebra. This is the genesis of the homomorphism Θ.

Proposition 6.4 *The simple representations of* $\mathbf{SO}(3)$ *are the representations* D_j *for integral* j*:*

$$D_0 = 1,\ D_1,\ D_2, \ldots$$

Proof ▶ We have established that the simple representations of $\mathbf{SO}(3)$ are just those D_j which are trivial on $\{\pm I\}$. But under $-I$,

$$(z,w) \mapsto (-z,-w)$$

and so if $P(z,w)$ is a homogeneous polynomial of degree $2j$,

$$P(-z,-w) = (-1)^{2j}P(z,w).$$

Thus $-I$ acts trivially on V_j if and only if $2j$ is even, ie j is integral. ◀

The following result is almost obvious.

Proposition 6.5 *Let ρ be the natural representation of $\mathbf{SO}(3)$ is $\mathbb{R}^3$. Then*

$$\mathbb{C}\rho = D_1.$$

Proof ► To start with, ρ is simple. For if it were not, it would have a 1-dimensional sub-representation. In other words, we could find a direction in $\mathbb{R}^3$ sent into itself be every rotation, which is absurd.

It follows that $\mathbb{C}\rho$ is simple. For otherwise it would split into 2 conjugate parts, which is impossible since its dimension is odd.

The result follows since D_1 is the only simple representation of dimension 3. ◄

Chapter 7

The Peter-Weyl Theorem

7.1 The finite case

Suppose G is a finite group. Recall that

$$C(G) = C(G, \mathbb{C})$$

denotes the banach space of maps $f : G \to \mathbb{C}$, with the norm

$$|f| = \sup_{g \in G} |f(g)|.$$

(For simplicity we restrict ourselves to the case of complex scalars: $k = \mathbb{C}$.)

The group G acts on $C(G)$ on both the left and the right. These actions can be combined to give an action of $G \times G$:

$$((g, h)f)(x) = f(g^{-1}xh).$$

Recall that the corresponding representation τ of $G \times G$ splits into simple parts

$$\tau = \sigma_1 * \times \sigma_1 + \cdots + \sigma_s * \times \sigma_s$$

where $\sigma_1, \ldots, \sigma_s$ are the simple representations of G (over $\mathbb{C}$).

Suppose V is a G-space. We have a canonical isomorphism

$$\hom(V, V) = V^* \otimes V.$$

Thus $G \times G$ acts on $\hom(V, V)$, with the first factor acting on V^* and the second on V. A little thought shows that this action can be defined as follows. Suppose $t : V \to V$ is a linear map, ie an element of $\hom(V, V)$. Then

$$(g, h)t = t'$$

where t' is the linear map
$$t'(v) = ht(g^{-1}v).$$

The expression for τ above can be re-written as

$$C(G) \equiv \hom(V_{\sigma_1}, V_{\sigma_1}) + \cdots + \hom(V_{\sigma_s}, V_{\sigma_s}),$$

where V_σ is the space carrying the simple representation σ.

In other words
$$C(G) = C(G)_{\sigma_1} \oplus \cdots \oplus C(G)_{\sigma_s},$$

where
$$C(G)_\sigma \equiv \hom(V_\sigma, V_\sigma).$$

Since the representations $\sigma^* \times \sigma$ of $G \times G$ are simple and distinct, it follows that the subspaces $C(G)_\sigma \subset C(G)$ are the isotypic components of $C(G)$.

If we pass to the (perhaps more familiar) regular representation of G in $C(G)$ by restricting to the subgroup $e \times G \subset G \times G$, so that G acts on $C(G)$ by

$$(gf)(x) = f(g^{-1}x),$$

then each subspace $V^* \otimes V$ is isomorphic (as a G-space) to $\dim \sigma V$. Thus it remains isotypic, while ceasing (unless $\dim \sigma = 1$) to be simple. It follows that the expression
$$C(G) = C(G)_{\sigma_1} \oplus \cdots \oplus C(G)_{\sigma_s},$$

can equally well be regarded as the splitting of the G-space $C(G)$ into its isotypic parts.

Whichever way we look at it, we see that each function $f(x)$ on G splits into components $f_\sigma(x)$ corresponding to the simple representations σ of G.

What exactly *is* this component $f_\sigma(x)$ of $f(x)$? Well, recall that the projection π of the G-space V onto its σ-component V_σ is given by

$$\pi = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

It follows that
$$\boxed{f_\sigma(x) = \frac{1}{|G|} \sum_{g \in G} \chi(g) f(gx).}$$