Resource B

Projective Geometry

B.1 Affine space

By the *affine plane* we mean the familiar plane from school geometry, where each point is represented by coordinates $(x, y) \in \mathbb{R}^2$, with the understanding that we "forget" the origin and coordinate axes. That is, if we choose a different origin and coordinates (x', y') we regard it as the same affine plane, with an *affine transformation*

$$\begin{aligned} x' &= ax + by + e, \\ y' &= cx + dy + f \end{aligned}$$

taking one set of coordinates into the other. (Here $ad - bc \neq 0$, since otherwise the transformation would not be bijective.)

We can represent this transformation by the 3×3 matrix

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix},$$

since it is readily verified that composition of transformations corresponds to muliplication of matrices.

The affine transformations of the plane form the *affine group*.

We extend this definition to any field k, and any dimension $n > 0$. Thus the affine space $\mathbb{A}^n(k)$ is the space where each point can be represnted by coordinates $(x_1, \ldots, x_n) \in k^n$. We shall often refer to this, slightly innacurately, as the affine space k^n.

Note that we do not have any notion of distance or angle in affine geometry. We use the term *euclidean space* if distance is defined.

B.2 Projective space

Definition B.1 *If V is a vector space over the field k then the corresponding* projective space $\mathbb{P}V$ *is the set of 1-dimensional subspaces of V. In other words,*

$$\mathbb{P}V = (V \setminus \{0\})/k^\times.$$

The n-dimensional vector space over k is

$$\mathbb{P}^n(k) = \mathbb{P}(k^{n+1}).$$

Thus a point $P \in \mathbb{P}^n(k)$ is given by $n+1$ coordinates,

$$P = [x_1, \ldots, x_n, x_{n+1}],$$

with $x_1, \ldots, x_{n+1}$ not all zero, where a scalar multiple defines the same point:

$$[\lambda x_1, \ldots, \lambda x_{n+1}] = [x_1, \ldots, x_{n+1}].$$

Each non-singular linear map

$$t : k^{n+1} \to k^{n+1}$$

defines a *projective transformation*

$$T : \mathbb{P}^n(k) \to \mathbb{P}^n(k).$$

The maps $t, \lambda t$ (where $\lambda \in k^\times$) define the same projective transformation T. Thus the projective transformations form the projective group

$$\mathrm{PGL}(n, k) = \mathrm{GL}(n, k)/k^\times.$$

We can identify the affine plane $\mathbb{A}^2(k)$ with a subset of the projective plane $\mathbb{P}^2(k)$ through the map

$$(x, y) \mapsto (x, y, 1).$$

The points of the projective plane that are not in the affine plane, namely those of the form $(x, y, 0)$, form the *line at infinity* $z = 0$.

B.3 Curves

A *curve* Γ in the affine plane $\mathbb{A}^2(k)$ is defined by an irreducible polynomial

$$f(x, y) \in k[x, y].$$

(By irreducible we mean irreducible over the algebraic closure $\bar{k}$ of k.)

We associate to Γ the set of points (x, y) satisfying

$$f(x, y) = 0.$$

Proposition B.1 *An irreducible curve over k is completely determined by the points on it over $\bar{k}$.*

Remark: A curve over k need not have any points over k on it, eg the curve

$$x^2 + y^2 + 1 = 0$$

over $\mathbb{R}$ does not contain any real points.

B.4 The Erlanger Program

Felix Klein — perhaps the leading German mathematician between Gauss and Hilbert — suggested in a lecture in 1872 that geometry should be brought under the aegis of group theory. To each geometry — as, for example, affine geometry, projective geometry, spherical geometry, euclidean geometry, hyperbolic geometry — there corresponds a certain group of transformations. Two configurtations are regarded as identical if there is a transformation taking one into the other; and geometric problems reduce to problems about invariants of the particular group in question.

In our case there are 2 groups of interest, the affine group and the projective group, although later we shall add the group of *birational transformations* of a curve.

B.5 Geometry of the projective plane

In the projective plane, there is a duality between points and lines: any 2 lines define a point (where they meet), and any 2 points define a line.

As is readily verified, there is a unique projective transformation T taking any 4 non-collinear points A, B, C, D into any 4 non-collinear points A', B', C', D':

$$T(A) = A',\ T(B) = B',\ T(C) = C',\ T(D) = D'.$$

In particular, given 4 non-collinear points A, B, C, D, we can choose coordinates so that

$$A = [1, 0, 0],\ B = [0, 1, 0],\ C = [0, 0, 1],\ D = [1, 1, 1].$$

We cannot in general send 4 *collinear* points A, B, C, D into 4 collinear points A', B', C', D'. This is possible only if the sets of points have the same *cross-ratio*. (It is left to the student to find out what this means, and to verify the result.)

B.6 Tangents

Suppose

$$\gamma : f(x, y) = 0$$

is an irreducible curve in the affine plane. The tangent at a given point $P = (x_0, y_0)$ on γ is given by

$$y = mx + c,$$

where

$$m = \frac{dy}{dx} = \frac{\partial f/\partial x}{\partial f/\partial y},$$

and c is determined (once m is known) by

$$f(x, mx + c) = 0.$$

This turns out to be much simpler in the projective case. The tangent at $P = [X_0, Y_0, Z_0]$ on the curve

$$\Gamma : F(X, Y, Z) = 0$$

is given by

$$\frac{\partial F}{\partial X}X + \frac{\partial F}{\partial Y}Y + \frac{\partial F}{\partial Z}Z = 0,$$

where the partial derivatives are computed at P.

We say that the point P is *singular* if the tangent at P is undefined, ie if

$$\frac{\partial F}{\partial X} = 0, \ \frac{\partial F}{\partial Y} = 0, \ \frac{\partial F}{\partial Z} = 0.$$

We say that the curve Γ is *non-singular* if it contains no singular points, where as usual we must allow points defined over $\bar{k}$.

In general we shall be interested in non-singular curves, but which we understand that the curve is irreducible and non-singular.

One small point. The polynomial $F(X, Y, Z)$ is homogeneous, ie if F is of degree d then

$$F(tX, tY, tZ) = t^d F(X, Y, Z).$$

On differentiating with respect to t and then setting $t = 1$,

$$\frac{\partial F}{\partial X}X + \frac{\partial F}{\partial Y}Y + \frac{\partial F}{\partial Z}Z = dF(X, Y, Z).$$

This says, in effect, that the tangent at P passes through P.

Exercises 2 Projective Geometry

** 1. Show that there is an affine transformation taking a triangle ABC in the affine plane into any other triangle $A'B'C'$.
Is this transformation unique?

*** 2. Show that you can represent an affine transformation by a matrix

$$\begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix},$$

in the sense that the product of two affine transformations is represented by the product of the corresponding matrices.

** 3. Can you find an affine transformation taking 3 collinear (but distinct) points A, B, C into any other 3 collinear (but distinct) points A', B', C'.

*** 4. Give a construction in the affine plane to determine the mid-point of two points.

* 5. Find the projective line through the points $[1, 3, 3], [0, 1, 1]$.

* 6. Find the point in which the lines $x + y - z = 0,\ 2x - y = 0$ meet

** 7. Find a projective transformation taking the points $P = [1, 0, 0],\ Q = [0, 1, 0],\ R = [0, 0, 1],\ S = [1, 1, 1]$ into the points S, R, Q, P, in that order.

*** 8. Show that there is a projective transformation of the line $\mathbb{P}^1(k)$ taking any 3 points into any 3 points.
Is the transformation unique?
Find a transformation taking $0, 1, \infty$ into $1, \infty, 0$, in that order.

** 9. Define the *cross-ratio* $(A, B; C, D)$ of 4 points on a line. [Look up the definition.]
Show that the cross-ratio is preserved by projective transformations.

*** 10. Does there exist a projective transformation taking 2 lines l, m and 2 points P, Q not on either line into a similar configuration?
If so, is the transformation unique?

*** 11. Show that the projective place $\mathbb{P}^1(\mathbb{F}_p)$ contains $p^2 + p + 1$ points.
How many points are there on each line?

*** 12. How many points does the projective space $\mathbb{P}^n(\mathbb{F}_p)$ contain?

*** 13. How many projective planes does the projective 3-space $\mathbb{P}^3(\mathbb{F}_p)$ contain?

*** 14. What is the order of $\mathrm{PGL}(1, \mathbb{F}_p)$, ie how many projective transformations of the line $\mathbb{P}^1(\mathbb{F}_p)$ are there?

*** 15. What is the order of $\mathrm{PGL}(2, \mathbb{F}_p)$, ie how many projective transformations of the projective plane $\mathbb{P}^2(\mathbb{F}_p)$ are there?

**** 16. An (abstract) finite projective plane Π is a finite set of points, together with a set of subset of Π called *lines*, with the property that there is just one line containing any 2 points, and just one point on any 2 lines.

Show that the number of points on each line is the same, that the number of lines through each point is the same, and that these 2 numbers are equal.

**** 17. If there are q points on a line in the finite projective plane Π, how many points are there in Π?

**** 18. Is the projective plane $\mathbb{P}^2(\mathbb{F}_3)$ the only projective plane with 4 points on each line?

Are any two such planes isomorphic? [Explain what you mean by an isomorphism in this context.]

**** 19. Prove Pappus' Theorem, that if the points $\{A, B, C\}$ and $\{A', B', C'\}$ are collinear, then so are the meets $(AB', BA'),\ (BC', CB'),\ (CA', AC')$.

**** 20. Prove Desargue's Theorem, that if $ABC, A'B'C'$ are triangles in perspectivity (ie AA', BB', CC' are concurrent) then the 3 meets of corresponding sides $AB, A'B'$, etc, are collinear.