Resource H

The Modular Group

Recall that

$$\mathrm{SL}(2,\mathbb{R}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R},\ ad - bc = 1\}.$$

By analogy we set

$$\mathrm{SL}(2,\mathbb{Z}) = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z},\ ad - bc = 1\}.$$

Note that $\pm I \in \mathrm{SL}(2,\mathbb{Z})$.

Definition H.1 *The* modular group Γ *is the quotient-group*

$$\Gamma = \mathrm{SL}(2, R)/\{\pm I\}.$$

Thus each element $g \in \Gamma$ corresponds to two matrices $\pm X$.

The modular group Γ acts on the complex plane through the transforms

$$gz = \frac{az + b}{cz + d}.$$

(Note that the matrices $g(z) = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ define the same transform.)

Proposition H.1 *If* $g(z) = \frac{az+b}{cz+d}$ *then*

$$\Im(z) > 0 \Longrightarrow \Im(gz) > 0.$$

Proof ▶ If $z = x + iy$ then

$$\Im(z) = y = \frac{z - \bar{z}}{2i}$$

Thus

$$\begin{aligned}
\Im(gz) &= \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\bar{z}+b}{c\bar{z}+d}\right)\\
&= \frac{1}{2i}\left(\frac{(az+b)(c\bar{z}+d)-(a\bar{z}+b)(cz+d)}{(cz+d)(c\bar{z}+d)}\right)\\
&= \frac{1}{2i}\left(\frac{(ab-cd)(z-\bar{z})}{|cz+d|^2}\right)\\
&= \frac{1}{|cz+d|^2}\,\Im(gz).
\end{aligned}$$

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It follows that Γ acts on the upper half-plane

$$\mathcal{H} = \{z : \Im(z) > 0\};$$

and this is how the modular group is usually seen, as the group of transforms

$$z \mapsto gz = \frac{az+b}{cz+d} : \mathcal{H} \to \mathcal{H}.$$

Definition H.2 *We define $s, t \in \Gamma$ as the elements corresponding to the matrices*

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

In terms of transforms $\mathcal{H} \to \mathcal{H}$,

$$sz = -1/z,\ tz = z+1.$$

Proposition H.2 $s^2 = 1, (st)^3 = 1.$

Proof ▶ In the first place,

$$S^2 = -I \Longrightarrow s^2 = 1.$$

Secondly,

$$ST = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

has characteristic equation

$$\det(\lambda I - ST) = \lambda(\lambda-1)+1 = \lambda^2 - \lambda + 1.$$

It follows (from the Cayley-Hamilton theorem) that ST satisfies

$$\lambda^2 - \lambda + 1 = 0 \Longrightarrow \lambda^3 = -1.$$

(Of course one can show directly that $(ST)^3 = -I$.) ◀

Proposition H.3 Γ *is generated by s, t:*

$$\Gamma = \langle s, t \rangle.$$

Proof ▶ Suppose

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Our strategy is to act on X with S, T and T^{-1} on either side so as to steadily reduce $|a|$, $|b|$ and $|c|$ until $b = c = 0$ and $X = \pm I$. We implement this through the following algorithm.

Step 1 If $|d| < |a|$ then we pass to

$$SXS^{-1} = -SXS = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}.$$

Thus we may assume that $|a| \leq |d|$.

Step 2 If $a = 0$ then $|bc| = 1 \Longrightarrow |b| = |c| = 1$. In this case we pass to

$$SX = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} = \pm \begin{pmatrix} 1 & -d \\ 0 & 1 \end{pmatrix} = \pm T^{-d},$$

and we are done. Otherwise we may assume that $0 < |a| \leq |d|$.

Step 3 Since

$$XT^r = \begin{pmatrix} a & b + ra \\ c & d + rc \end{pmatrix}.$$

we can replace X by XT^r so that

$$|b + ra| \leq |a|/2.$$

Note that this step does not affect a or c.

If now $|d| < |a|$ we return to step 1. Otherwise we may assume that $|b| \leq |a|/2, \ |a| \leq |d|$.

Step 4 Note that, by the formula in Step 1,

$$ST^rS^{-1} = ST^rS = \begin{pmatrix} 1 & 0 \\ -r & 1 \end{pmatrix}.$$

Thus

$$ST^rSX = \begin{pmatrix} a & b \\ c - ra & d - rb \end{pmatrix},$$

and we can replace X by ST^rSX so that

$$|c - ra| \leq |a|/2,$$

without affecting a or b,

If now $|d| < |a|$ we return to step 1. Otherwise we may assume that $|b| \leq |a|/2,\ |c| \leq |a|/2$.

Step 5 Finally,

$$1 = |ad - bc| \geq |ad| - |a^2|/4 \geq |ad| - |ad|/4 \geq \frac{3}{4}|ad| \Longrightarrow |ad| \leq \frac{4}{3} \Longrightarrow |ad| = 1.$$

Thus

$$|a| = |d| = 1 \Longrightarrow b = c = 0 \Longrightarrow X = \pm I.$$

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Exercises 8 Modular Group

In exercises 1–3 express the given transform gz as the shortest possible word in $s,\ t$ and t^{-1}.

** 1. $f(z) = z - 2$.

** 2. $gz = \frac{1}{3-z}$.

*** 3. $gz = \frac{2z-3}{5z-7}$.

*** 4. Show that each $g \in \Gamma$ is uniquely expressible in the standard form

$$g = r^{i_0} s \dots s r^{i_{n-1}} s r^{i_n},$$

where $i_j \in \{\pm 1\}$ if $1 \le j < n,\ \ i_1, i_n \in \{0, \pm 1\}$.

In exercises 5–7 express the given transform in the above standard form.

** 5. $gz = z + 2$.

** 6. $gz = \frac{1}{1-z}$.

** 7. $gz = \frac{3z+5}{4z+7}$.

*** 8. Show that there are just two transforms $g \in \Gamma$ such that $g(i) = i$.

*** 9. Show that there are just three transforms $g \in \Gamma$ such that $g(\omega) = \omega$.

Let $F = \{z \in \mathcal{H} : \Re z \le 1/2,\ |z| \ge 1\}$, and let B be the part of the boundary of F with $\Re z > 0$.

**** 10. Show that every $z \in \mathcal{H}$ has a transform $gz \in F$.

**** 11. Show that every $z \in \mathcal{H}$ has a unique transform $gz \in F \setminus B$.

*** 12. Show that there are just 3 points $z \in F$ with non-trivial stability subgroup $\{g \in \Gamma : gz = z\}$.

*** 13. Show that $z \in \mathcal{H}$ has non-trivial stability group if and only if z is a transform of i or ω.

** 14. Show that the centre of $\mathrm{SL}(2, \mathbb{Z})$ is $\{\pm I\}$.