Resource G

Cauchy's Theorem and its consequences

Recall that the function $f(z)$ is said to be *holomorphic* in an open set $U \subset \mathbb{C}$ if it is differentiable at each point $z \in U$; while it is said to be *meromorphic* in U if it is either differentiable or has a pole of finite order at each point $z \in U$. ($f(z)$ is said to have a pole of order n at a if

$$f(z) = \frac{g(z)}{(z-a)^n}$$

where $g(z)$ is holomorphic in some open set $U \ni a$ with $g(a) \neq 0$.)

We recall some fundamental results from complex analysis:

Cauchy's Theorem *If the function $f(z)$ is holomorphic in the open set $U \subset \mathbb{C}$, and $C \subset U$ is a Jordan curve then*

$$\int_C f(z)dz = 0.$$

This is the fundamental result of complex analysis.

A *Jordan curve* is a continuous loop in $\mathbb{C}$ which does not intersect itself. In practice we will only use the simplest of curves, eg the perimeter of a circle or polygon, and in particular the perimeter of a fundamental parallelogram of an elliptic function.

By convention we always take the integral in the counter-clockwise direction around C.

In the following results, we shall always make the same assumptions, that $f(z)$ is holomorphic in the open set $U \subset \mathbb{C}$, and that $C \subset U$ is a Jordan curve.

Cauchy's Integral Formula *If a is inside C then*

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz,$$

Infinite differentiability With the same assumption,

$$f'(a) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-a)^2} dz,$$

and more generally,

$$f^{(n)}(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^n} dz,$$

These results are obtained by differentiating Cauchy's Integral Formula with respect to a under the integral sign.

It follows that if $f(z)$ is differentiable in an open set $U \subset \mathbb{C}$ then it is differentiable infinitely often in U.

The Residues Theorem Suppose $f(z)$ has a pole of order n at $z = b$, so that it has an expansion

$$f(z) = \frac{c_{-n}}{(z-b)^n} + \cdots + \frac{c_{-1}}{z-b} + c_0 + \cdots$$

in a neighbourhood of b. Then the *residue* of $f(z)$ at b is defined to be c_1.

Suppose $f(z)$ has poles at $b_1, b_2, \ldots, b_r$ inside C, with residues $c_1, c_2, \ldots, c_r$. Then

$$\frac{1}{2\pi i} \int_C f(z) dz = c_1 + c_2 + \cdots + c_r.$$

Liouville's Theorem *If $f(z)$ is holomorphic and bounded in the whole of $\mathbb{C}$ then it is constant.*

This follows on taking C to be a large circle of radius R, giving

$$|f'(a)| \leq \frac{1}{2\pi} \frac{2\pi R}{R^2} = \frac{c}{R}$$

if $|f(z)| \leq c$. Since R is arbitrary it follows that $f'(a) = 0$ for all a, and so $f(z)$ is constant.

Counting poles and zeros *Suppose $f(z)$ has zeros at $a_1, a_2, \ldots, a_r$ and poles at $b_1, b_2, \ldots, b_s$ inside C; and suppose $f(z)$ has no poles or zeros on C. Then*

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = r - s.$$

Here it is understood that that poles and zeros are counted with appropriate multiplicity, eg a double zero is counted twice.

The result follows from the fact that the function $f'(z)/f(z)$ has a simple pole with residue d at a zero of order d, and a simple pole with residue $-d$ at a pole of order d.

Addition Theorem

$$\frac{1}{2\pi i}\int_C z\frac{f'(z)}{f(z)}dz = (a_1 + \cdots + a_r) - (b_1 + \cdots + b_s).$$

For if $f(z)$ has a zero at a of order m then $zf'(z)/f(z)$ has a simple pole at a with residue ma; while if $f(z)$ has a pole at b of order n then $zf'(z)/f(z)$ has a simple pole at b with residue $-nb$.

Uniform convergence *If each of the functions $u_n(z)$ is holomorphic in the open set $U \subset \mathbb{C}$ and $\sum u_n(z)$ is uniformly convergent in U then*

$$f(z) = \sum u_n(z)$$

is holomorphic in U, with

$$f'(z) = \sum u_n'(z).$$

Notice that this is much simpler to prove than the corresponding result for real functions, using the fact that

$$f(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz,$$

With the same assumptions, *if C is a contour inside U then*

$$\int_C f(z)dz = \sum \int_C u_n(z)dz.$$

Exercises 7 Discriminant

In exercises 1–5 determine the poles of the given function and the residues at the poles.

** 1. $f(z) = \frac{z^2-1}{z^2+1}$

** 2. $f(z) = \tan z$

** 3. $f(z) = \cot z$

** 4. $f(z) = \frac{1}{z^4-1}$

** 5. $f(z) = \frac{z^3}{2z^2-i}$

In exercises 6–10 Determine the integral of the given function around the unit circle.

** 6. $\tan z$

** 7. $\cot z$

** 8. $z \cot z$

** 9. $f(z) = \frac{4z}{2z^1-1}$

** 10. $f(z) = \frac{e^{2z}}{2z-1}$

*** 11. Determine $\int_C \frac{dz}{(z-z_1)(z-z_2}$ if z_1, z_2 lie within C.

*** 12. Show that if the function $f(z)$ is holomorphic in the circle $|z| < R$ then it has a power-series expansion valid in this region.

*** 13. If the polynomial

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

satisfies the inequality $|f(x)| \le M$ on the unit circle $|z| = 1$, show that $|a_i| \le M$ for $i = 1, \ldots, n$.

*** 14. Given that $f(z) = z^2$ on the unit circle, determine its value inside the circle.

*** 15. Show that if $f(z)$ is homomorphic in $\mathbb{C}$, and satisfies $|f(z)| \le |z^n|$ at each point, then $f(z)$ is a polynomial.