Course MA346H Sample Exam Paper 2

Dr Timothy Murphy

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1. Outline Chaitin's definition of a Turing machine T.

What is meant by saying that a set of strings (of 0's and 1's) is *prefix-free*? Show that the set

$$\{p: T(p) \text{ defined}\}\$$

is prefix-free.

Define a prefix-free code [n] for the natural numbers $n \in \mathbb{N}$; and sketch the construction of a Turing machine T such that

$$T([m][n]) = [mn].$$

Do there exist maps $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which cannot be implemented by a Turing machine in this way?

2. Define the algorithmic entropy H(s) of a string s (of 0's and 1's). Show that there exists a number N such that

$$H(s) \le |s| + 2\log_2|s|$$

for all strings s with $|s| \ge N$. (Here |s| denotes the length of the string s.)

Show conversely that there exist an infinity of strings s with

$$H(s) \ge |s| + \log_2 |s|.$$

- 3. What is meant by saying that two sets X, Y have the same cardinality? Show that the cardinality of a set X is strictly less than the cardinality of the set 2^X of subsets of X.
- 4. What is meant by saying that a set of strings (of 0's and 1's) is (a) *recursive*, and (b) *recursively enumerable*?

Show that every recursive set is recursively enumerable, but that the converse is not true: there exists a recursively enumerable set that is not recursive.