Course MA346H
Sample Exam Paper 2

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1. Outline Chaitin's definition of a Turing machine T.

   What is meant by saying that a set of strings (of 0's and 1's) is *prefix-free*? Show that the set

   $$\{p : T(p) \text{ defined}\}$$

   is prefix-free.

   Define a prefix-free code $[n]$ for the natural numbers $n \in \mathbb{N}$; and sketch the construction of a Turing machine T such that

   $$T([m][n]) = [mn].$$

   Do there exist maps $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ which cannot be implemented by a Turing machine in this way?

2. Define the algorithmic entropy $H(s)$ of a string s (of 0's and 1's).

   Show that there exists a number N such that

   $$H(s) \leq |s| + 2 \log_2 |s|$$

   for all strings s with $|s| \geq N$. (Here $|s|$ denotes the length of the string s.)

   Show conversely that there exist an infinity of strings s with

   $$H(s) \geq |s| + \log_2 |s|.$$

   Answer:

   (a) *Suppose T is a Turing machine. We set*

   $$H_T(s) = \min_{p : T(p) = s} |s|;$$

and we set

$$H(s) = H_U(s),$$

where U is our chosen universal machine.

In other words, $H(s)$ is the length of the shortest string p which when input into U will output s.

Suppose we chose another universal machine V in place of U. By the definition of a universal machine, there exist strings u, v such that

$$U(vs) = V(s), \quad V(us) = U(s).$$

It follows that

$$H_U(s) \le H_V(s) + |v|, \quad H_V(s) \le H_U(s) + |u|.$$

Thus

$$H_V(s) = H_U(s) + O(1).$$

(b) Let $B(n)$ denote the binary code for $n \in \mathbb{N}$, eg

$$B(5) = 101,\ B(12) = 1100.$$

Given a string s of length n, let us code s by

$$[s] = \langle B(n) \rangle s,$$

where $\langle B(n) \rangle$ is the code for the string $B(n)$, eg

$$\langle 1100 \rangle = 111110100$$

(with the final 0 signalling the end of the string).

It is clear that we can construct a Turing machine T which will output s when $[s]$ is input; it first decodes $\langle B(n) \rangle$, and then outputs the next n bits.

Thus

$$\begin{aligned} H_T(s) &\le n + |\langle B(n) \rangle| \\ &\le |s| + 2\log_2 n + 3, \end{aligned}$$

since

$$|B(n)| \le \log_2 n + 1.$$

It follows that

$$H(s) \le |s| + 2\log_2 n + 3 + |\langle T \rangle|.$$

This doesn't quite give the required result. However, we could replace $\langle Bn \rangle s$ by

$$\langle B(B(n)) \rangle B(n) s,$$

so the machine T has first to decode $B(B(n))$, and work out how many bits there are in $B(n)$, then determine n, and finally output s.
This gives

$$H(s) \leq |s| + \log(\log_2 n + 3) + \log_2 n + 2 + |\langle T \rangle|,$$

and so

$$H(s) \leq |s| + 2\log_2 n$$

for sufficiently large n.
[The argument shows in fact that for sufficiently large n,

$$H(s) < n + \log n + \log\log n + \cdots,$$

where we continue the sum for any finite number of terms.

(c) *We know by Kraft's Inequality that*

$$\sum 2^{-H(s)} \leq 1,$$

since the inputs p for which $U(p)$ is defined form a prefix-free set.
Suppose that in fact

$$H(s) < |s| + \log_2 |s|$$

for all sufficiently long strings s. Then

$$2^{-H(s)} > 2^{-(n+\log_2 n)} = 2^{-n}/n$$

for each such string.
Since there are 2^n strings of length n, these will together contribute $> 1/n$ to the sum. Since

$$\sum \frac{1}{n}$$

is divergent, the result follows.

3. What is meant by saying that two sets X, Y have the same cardinality?

   Show that the cardinality of a set X is strictly less than the cardinality of the set 2^X of subsets of X.

4. What is meant by saying that a set of strings (of 0's and 1's) is (a) *recursive*, and (b) *recursively enumerable*?

   Show that every recursive set is recursively enumerable, but that the converse is not true: there exists a recursively enumerable set that is not recursive.