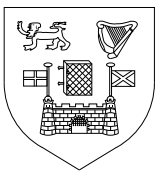

Course 424 — Group Representations

Sample Exam

Dr Timothy Murphy

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Attempt 7 questions. (If you attempt more, only the best 7 will be counted.) All questions carry the same number of marks.
In this paper representation *means "finite-dimensional representation over* $\mathbb{C}$*".*

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Find all representations of S_3 of degree 2 (up to equivalence).

   What is meant by saying that a representation α is *simple*? Find all simple representations of D_4 from first principles.

   Answer:

   (a) *A representation α of a group G in a vector space V is a homomorphism*

   $$\alpha : G \to \mathbf{GL}(V).$$

   (b) *The representations α, β of G in U, V respectively are said to be equivalent if U, V are isomorphic as G-spaces, that is, there exists an invertible map*

   $$t : U \to V$$

   which preserves the action of G, ie

   $$t(gu) = g(tu)$$

   for all $g \in G, u \in U$.

   (c) *Suppose α is a representation of*

   $$S_3 = \langle s, t : s^3 = t^2 = 1, st = ts^2 \rangle$$

   (taking $s = (abc), t = (ab)$) in the 2-dimensional vector space V.

Let e be an eigenvector of s with eigenvalue λ:

$$se = \lambda e.$$

Since $s^3 = 1$, $\lambda \in \{1, \omega, \omega^2\}$ (where $\omega = e^{2\pi/3}$).
Let $f = te$. Then

$$sf = ste = ts^2e = \lambda^2 te = \lambda^2 f.$$

In other words, f is also an eigenvector of s, but with eigenvalue λ^2.
If $\lambda^2 \neq \lambda$, ie $\lambda \neq 1$, then e, f are eigenvectors of s with different eigenvalues, and so are necessarily linearly independent. Thus in this case e, f is a basis for V, and the representation is completely determined. If $\lambda = \omega$ then

$$s \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad t \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $\lambda = \omega^2$ then f has eigenvalue $(\omega^2)^2 = \omega$, and we get the same representation with e, f interchanged. Thus there is just one representation in this case.
We must fall into this case unless both *the eigenvalues of s are 1. In the latter case, since $\alpha(g)$ is always semisimple (diagonalisable), we must have $s \to I$ with respect to any basis.*
But then we can diagonalise t without effecting s. Since $t^2 = 1$,

$$t \to \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Two of these representations are equivalent; so we get just 3 representations (in effect $1+1, 1+\epsilon, \epsilon+\epsilon$) in this case:

$$t \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus S_3 has just 4 representations of degree 2.

(d) *The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if*

$$g \in G, u \in U \implies gu \in U.)$$

(e) *We have*

$$D_4 = \langle t, s : s^4 = t^2 = 1, st = ts^3 \rangle.$$

Let us first suppose α is a 1-dimensional representations of D_4. ie a homomorphism

$$\alpha : D_4 \to \mathbb{C}^*.$$

Suppose

$$\alpha(s) = \lambda, \ \alpha(t) = \mu.$$

Then

$$\lambda^4 = \mu^2 = 1,\ \lambda\mu = \mu\lambda^3.$$

The last relation gives

$$\lambda^2 = 1.$$

Thus there are just 4 1-dimensional representations given by

$$s \mapsto \pm 1, t \mapsto \pm 1.$$

Now suppose α is a simple representation of G in the vector space V over $\mathbb{C}$, where $\dim V \geq 2$. Let $e \in V$ be an eigenvector of s:

$$se = \lambda e;$$

and let

$$f = te.$$

Then

$$sf = ste = ts^3e = \lambda^3 te = \lambda^3 f,$$

ie f is a λ^3-eigenvector of s.
It follows that the subspace

$$\langle e, f\rangle \subset V$$

is stable under D_4, since

$$se = \lambda e,\ sf = \lambda^3 f,\ te = f,\ tf = t^2 e = e.$$

Since V by definition is simple, it follows that

$$V = \langle e, f\rangle.$$

In particular, $\dim \alpha = 2$, and e, f form a basis for V.
Since $s^4 = 1$ we have $\lambda^4 = 1$, ie $\lambda \in \{\pm 1, \pm i\}$.
If $\lambda = 1$ then s would have eigenvalues $1, 1$ (since $1^3 = 1$). But we know that s (ie $\alpha(s)$) is diagonalisable. It follows that $s = I$.
Similarly if $\lambda = -1$ then s has eigenvalues $-1, -1$ and so $s = -I$.
In either of these cases s will always be diagonal. Since we can always diagonalise t, we can diagonalise s, t simultaneously. But in that case the representation would not be simple; for if e is a common eigenvector of s, t then the 1-dimensional space $\langle e\rangle$ is stable under D_4.
Thus we are left with the cases $\lambda = \pm i$. If $\lambda = -i$ then on swapping e and f we would have $\lambda = i$. So we have only 1 2-dimensional representation (up to equivalence):

$$s \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

In conclusion, D_4 has just 5 simple representations: 4 of dimension 1, and 1 of dimension 2.

2. What is meant by saying that a representation α is *semisimple*?

   State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

   Prove that every representation of a compact group is semisimple.

   Answer:

   (a) *The representation α of G in V is said to be semisimple if it can be expressed as a sum of simple representations:*

   $$\alpha = \sigma_1 + \cdots + \sigma_m.$$

   This is equivalent to the condition that each stable subspace $U \subset V$ has a stable complement W:

   $$V = U \oplus W.$$

   (b) *Haar's theorem states that there exists an invariant measure on a compact group G, unique up to a scalar multiple.*
   A measure *μ on X is a continuous linear functional*

   $$\mu : C(X) \to \mathbb{C},$$

   where $C(X) = C(X, \mathbb{R})$ is the space of real-valued continuous functions on X with norm $\|f\| = \sup |f(x)|$.
   The compact group G acts on $C(G)$ by

   $$(gf)(x) = f(g^{-1}x).$$

   The measure μ is said to be invariant under G if

   $$\mu(gf)\mu(f)$$

   for all $g \in G,\ f \in C(G)$.

   (c) *We can prove the existence of such an invariant measure in the following way.*
   By an average *F of $f \in C(G)$ we mean a function of the form*

   $$F = \lambda_1 g_1 f + \lambda_2 g_2 f + \cdots + \lambda_r g_r f,$$

   ie

   $$F(x) = \lambda_1 f(g_1^{-1}x) + \lambda_2 f(g_2^{-1}x) + \cdots + \lambda_r f(g_r^{-1}x).$$

   where $0 \leq \lambda_i \leq 1,\ \sum \lambda_i = 1$ and $g_1, g_2, \ldots, g_r \in G$.
   If F is an average of f then

   i. $\inf f \leq \inf F \leq \sup F \leq sup f$;
   ii. If μ is an invariant measure then $\mu(F) = \mu(f)$;
   iii. An average of F is an average of f.

Let us set

$$\mathrm{var}(f) = \sup f - \inf f$$

so that

$$\mathrm{var}(F) \leq \mathrm{var}(f)$$

for any average F of f.

We shall establish a sequence of averages $F_0 = f, F_1, F_2, \ldots$ (each an average of its predecessor) such that $\mathrm{var}(F_i) \to 0$. It will follow that

$$F_i \to c \in \mathbb{R},$$

ie $F_i(g) \to c$ for each $g \in G$.

Suppose $f \in C(G)$. It is not hard to find an average F of f with $\mathrm{var}(F) < \mathrm{var}(f)$. Let

$$V = \{g \in G : f(g) < \frac{1}{2}(\sup f + \inf f),$$

ie V is the set of points where f is 'below average'. Since G is compact, we can find $g_1, \ldots, g_r$ such that

$$G = g_1 V \cup \cdots \cup g_r V.$$

Consider the average

$$F = \frac{1}{r}\left(g_1 f + \cdots + g_r f\right).$$

Suppose $x \in G$. Then $x \in g_i V$ for some i, ie

$$g_i^{-1} x \in V.$$

Hence

$$(g_i f)(x) = f(g_i^{-1} x) < \frac{1}{2}(\sup f + \inf f),$$

and so

$$\begin{aligned} F(x) &< \frac{r-1}{r} \sup f + \frac{1}{2r}(\sup f + \inf f) \\ &= \sup f - \frac{1}{2r} \sup f - \inf f. \end{aligned}$$

Hence $\sup F < supf$ and so

$$\mathrm{var}(F) < \mathrm{var}(f).$$

This allows us to construct a sequence of averages $F_0 = f, F_1, F_2, \ldots$ such that

$$\mathrm{var}(f) = \mathrm{var}(F_0) > \mathrm{var}(F_1) > \mathrm{var}(F_2) > \cdots .$$

But that is not sufficient to show that $\mathrm{var}(F_i) \to 0$. For that we must use the fact that any $f \in C(G)$ is uniformly *continuous.*

[I would accept this last remark as sufficient in the exam, and would not insist on the detailed argument that follows.]

In other words, given $\epsilon > 0$ *we can find an open set* $U \ni e$ *such that*

$$x^{-1}y \in U \implies |f(x) - f(y)| < \epsilon.$$

Since

$$(g^{-1}x)^{-1}(g^{-1}y) = x^{-1}y,$$

the same result also holds for the function gf*. Hence the result holds for any average* F *of* f*.*

Let V *be an open neighbourhood of* e *such that*

$$VV \subset U, \quad V^{-1} = V.$$

(If V *satisfies the first condition, then* $V \cap V^{-1}$ *satisfies both conditions.) Then*

$$xV \cup yV \neq \emptyset \implies |f(x) - f(y)| < \epsilon.$$

For if $xv = yv'$ *then*

$$x^{-1}y = vv'^{-1} \in U.$$

Since G *is compact we can find* $g_1, \ldots, g_r$ *such that*

$$G = g_1V \cup \cdots \cup g_rV.$$

Suppose f *attains its minimum* $\inf f$ *at* $x_0 \in g_iV$*; and suppose* $x \in g_jV$*. Then*

$$g_i^{-1}x_0,\ g_j^{-1}x \in V.$$

Hence

$$\left(g_j^{-1}x\right)^{-1}\left(g_i^{-1}x_0\right) = \left(g_ig_j^{-1}x\right)^{-1}x_0 \in U,$$

and so

$$|f(g_ig_j^{-1}x) - f(x_0)| < \epsilon.$$

In particular,

$$(g_jg_i^{-1}f)(x) < \inf f + \epsilon.$$

Let F *be the average*

$$F = \frac{1}{r^2}\sum_{i,j} g_jg_i^{-1}f.$$

Then

$$\sup F < \frac{r^2-1}{r^2}\sup f + \frac{1}{r^2}(\inf f + \epsilon),$$

and so

$$\begin{aligned}\operatorname{var}(F) &< \frac{r^2-1}{r^2}\operatorname{var}(f) + \frac{1}{r^2}\epsilon \\ &< \frac{r^2-1/2}{r^2}\operatorname{var}(f),\end{aligned}$$

if $\epsilon < \mathrm{var}(f)/2$.
Moreover this result also holds for any average of f *in place of* f. *It follows that a succession of averages of this kind*

$$F_0 = f, F_1, \ldots, F_s$$

will bring us to

$$\mathrm{var}(F)_s < \frac{1}{2}\,\mathrm{var}(f).$$

Now repeating the same argument with F_s, *and so on, we will obtain a sequence of successive averages* $F_0 = f, F_1, \ldots$ *with*

$$\mathrm{var}(F)_i \downarrow 0.$$

It follows that

$$F_i \to c$$

(the constant function with value c*).*
It remains to show that this limit value c *is unique. For this we introduce right averages*

$$H(x) = \sum_j \mu_j f(xh_j)$$

where $0 \leq \mu_j \leq 1$, $\sum \mu_j = 1$. *(Note that a right average of* f *is in effect a left average of* $\tilde{f}$, *where* $\tilde{f}(x) = f(x^{-1})$. *In particular the results we have established for left averages will hold equally well for right averages.)*
Given a left average and a right average of f, *say*

$$F(x) = \sum \lambda_i f(g_i^{-1}x), \quad H(x) = \sum \mu_j f(xh_j),$$

we can form the joint average

$$J(x) = \sum_{i,j} \lambda_i \mu_j f(g_i^{-1}xh_j).$$

It is easy to see that

$$\inf F \leq \inf J \leq \sup J \leq \sup H,$$
$$\sup F \geq \sup J \geq \inf J \geq \inf H.$$

But if now $H_0 = f, H_1, \ldots$ *is a succession of right averages with* $H_i \to d$ *then it follows that*

$$c = d.$$

In particular, any two convergent sequences of successive left averages must tend to the same limit. We can therefore set

$$\mu(f) = c.$$

Thus $\mu(f)$ is well-defined; and it is invariant since f and gf have the same set of averages. Finally, if $f = 1$ then $\mathrm{var}(f) = 0$, and $f, f, f, \ldots$ converges to 1, so that

$$\mu(1) = 1.$$

The invariant measure on G is unique up to a scalar multiple. In other words, it is unique if we normalise the measure by specifying that

$$\mu(1) = 1$$

(where 1 on the left denotes the constant function 1).

(d) Suppose α is a representation of the finite group G in the vector space V. Let

$$P(u, v)$$

be a positive-definite hermitian form on V. Define the hermitian form Q on V by

$$Q(u, v) = \frac{1}{\|G\|} \sum_{g \in G} H(gu, gv).$$

Then Q is positive-definite (as a sum of positive-definite forms). Moreover Q is invariant under G, ie

$$Q(gu, gv) = Q(u, v)$$

for all $g \in G, u, v \in V$. For

$$\begin{aligned} Q(hu, hv) &= \frac{1}{\|G\|} \sum_{g \in G} H(ghu, ghv) \\ &= \frac{1}{|G|} \sum_{g \in G} H(gu, gv) \\ &= Q(u, v), \end{aligned}$$

since gh runs over G as g does.
Now suppose U is a stable subspace of V. Then

$$U^{\perp} = \{v \in V : Q(u, v) = 0 \forall u \in U\}$$

is a stable complement to U.
Thus every stable subspace has a stable complement, ie the representation is semisimple.

3. Define the *character* χ_α of a representation α, and show that it is a class function (ie it is constant on conjugacy classes).

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β of a group G, and show that if G is compact then

$$I(\alpha, \beta) = \int_G \overline{\chi_\alpha(g)} \chi_\beta(g)\, dg.$$

   Prove that a representation α is simple if and only if $I(\alpha, \alpha) = 1$.

   Answer:

(a) *Suppose α is a representation of the group G in the vector space V over k. Then the* character of α is the function

$$\chi : G \to k$$

given by

$$\chi(g) = \mathbf{tr}\alpha(g).$$

(b) Suppose $g \sim h$, say

$$h = xgx^{-1}.$$

Then

$$\alpha(h) = \alpha(x)\alpha(g)\alpha(x)^{-1}.$$

Hence

$$\chi(h) = \mathbf{tr}(\alpha(h)) = \mathbf{tr}(alpha(g)) = \chi(g),$$

since

$$\mathbf{tr}(TAT^{-1}) = \mathbf{tr}(T^{-1}TA) = \mathbf{tr}(A).$$

Thus χ is constant on conjugacy classes.

(c) The intertwining number $I(\alpha, \beta)$ of two representations $\alpha, \beta)$ of G in U, V respectively is the natural number

$$I(\alpha, \beta) = \dim \hom^G(U, V),$$

where $\hom^G(U, V)$ is the space of linear maps $f : U- > V$ stable under G, ie

$$f(gu) = gf(u)$$

for all $g \in G, u \in U$.

(d) The first step in proving the formula is to show that it is sufficient to consider the case $\alpha = 1$.
For on the left-hand side

$$I(\alpha, \beta) = I(1, \alpha^*\beta).$$

This follows from the canonical isomorphism

$$U^* \otimes V \to \hom(U, V),$$

where $(\pi, v) \mapsto \pi(v)$.
It is easy to see that this isomorphism preserves the action of G, so that

$$\hom^G(U, V) = (U^* \otimes V)^G = \hom^(\, k, U^* \otimes V),$$

from which the result follows on taking dimensions.
And on the right-hand side we have

$$\begin{aligned}\int_G \overline{\chi_\alpha(g)}\chi_\beta(g)\, dg &= \int_G \chi_{\alpha^*}(g)\chi_\beta(g)\, dg \\ &= \int_G \chi_{\alpha^*\beta}(g)\, dg \\ &= \int_G \chi_1(g)\chi_{\alpha^*\beta}(g)\, dg.\end{aligned}$$

Thus we have to show that

$$I(1,\alpha) = \int_G \chi(g)dg.$$

Now (as we just saw)

$$\hom^(\, k, U) = U^G,$$

the subspace formed by the elements of U invariant under G. Thus we have to show that

$$\dim U^G = \int_G \chi(g)dg.$$

Consider the linear transformation

$$\Pi = \int \alpha(g)dg.$$

is a projection onto the subspace U^G, ie $\Pi^2 = \Pi$. For if v is fixed under G then $\Pi(v) = v$; while for any v, $\Pi(v)$ (the average of the transforms of v) is fixed under G.

But if $\pi : V \to V$ is a projection onto a subspace $U \subset V$ then

$$\mathbf{tr}\pi = \dim U.$$

For we can construct a basis for V by combining a basis for $U = \mathbf{im}\pi$ and $W = \ker \pi$; and the matrix of Π with respect to this basis will be the diagonal matrix $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ with $\dim U$ 1's.

The result now follows since

$$\mathbf{tr}(\Pi) = \int \mathbf{tr}(alpha(g))dg = \int \chi_\alpha(g)dg.$$

4. Draw up the character table for S_4.

   Determine also the representation-ring for this group, ie express the product $\alpha\beta$ of each pair of simple representation as a sum of simple representations.

   Answer:

   (a) S_4 has 5 classes, corresponding to the types $1^4, 1^2 2, 13, 2^2, 4$. Thus S_4 has 5 simple representations.

   Each symmetric group S_n (for $n \geq 2$) has just 2 1-dimensional representations, the trivial representation 1 and the parity representation ϵ.

   Let $S_4 = \mathbf{Perm}(X)$, where $X = \{a, b, c, d\}$. The action of S_4 on X defines a 4-dimensional representation ρ of S_4, with character

$$\chi(g) = |\{x \in X : gx = x\}|$$

   In other words $\chi(g)$ is just the number of 1-cycles in g.

So now we can start our character table (where the second line gives the number of elements in the class):

	1^4	1^22	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
ρ	4	2	1	0	0

Now

$$I(\rho,\rho)=\frac{1}{24}(1\cdot 16+6\cdot 4+8\cdot 1)=2.$$

It follows that ρ has just 2 simple parts. Since

$$I(1,\rho)=\frac{1}{24}(1\cdot 4+6\cdot 2+8\cdot 1)=1,$$

It follows that

$$\rho=1+\alpha,$$

where α is a simple 3-dimensional representation, with character given by

$$\chi(g)=\chi_\rho(g)-1.$$

The representation $\epsilon\alpha$ is also simple, and is not equal to α since it has a different character. So now we have 4 simple characters of S_4, as follows:

	1^4	1^22	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1

To find the 5th simple representation, we can consider α^2. This has character

	1^4	1^22	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
α^2	9	1	0	1	1

We have

$$I(1,\alpha^2)=\frac{1}{24}(9+6+3+6)=1,$$
$$I(\epsilon,\alpha^2)=\frac{1}{24}(9-6+3-6)=0,$$
$$I(\alpha,\alpha^2)=\frac{1}{24}(27+6-3-6)=1,$$
$$I(\epsilon\alpha,\alpha^2)=\frac{1}{24}(27-6-3+6)=1.I(\alpha^2,\alpha^2)\ =\frac{1}{24}(81+6+3+6)=4,$$

It follows that α^2 has 4 simple parts, so that

$$\alpha^2 = 1 + \alpha + \epsilon\alpha + \beta,$$

where β is the 5th simple representation, with character given by

$$\chi_\beta(g) = \chi_\alpha(g)^2 - 1 - \chi_\alpha(g) - \epsilon(g)\chi_\alpha(g).$$

This allows us to complete the character table:

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1
β	2	0	-1	2	0

(b) *We already know how to express α^2 in terms of the 5 simple representations. Evidently $\epsilon\beta = \beta$ since there is only 1 simple representation of dimension 2. The character of $\alpha\beta$ is given by*

	1^4	$1^2 2$	13	2^2	4
$\alpha\beta$	6	0	0	-2	0

We have

$$I(\alpha\beta, \alpha\beta) = \frac{1}{24}(36 + 12) = 2.$$

Thus $\alpha\beta$ has just 2 simple parts. These must be α and $\epsilon\alpha$ to give dimension 6:

$$\alpha\beta = \alpha + \epsilon\alpha.$$

Also we have

$$I(\beta^2, \beta^2) = \frac{1}{24}(16 + 8 + 48) = 3.$$

Thus β has 3 simple parts. So by dimension, we must have

$$\beta^2 = 1 + \epsilon + \beta.$$

Now we can give the multiplication table for the representation-ring:

	1	ϵ	β	α	$\epsilon\alpha$
1	1	ϵ	β	α	$\epsilon\alpha$
ϵ	ϵ	1	β	$\epsilon\alpha$	α
β	β	β	$1 + \epsilon + \beta$	$\alpha + \epsilon\alpha$	$\alpha + \epsilon\alpha$
α	α	$\epsilon\alpha$	$\alpha + \epsilon\alpha$	$1 + \beta + \alpha + \epsilon\alpha$	$\epsilon + \beta + \alpha + \epsilon\alpha$
$\epsilon\alpha$	$\epsilon\alpha$	α	$\alpha + \epsilon\alpha$	$\epsilon + \beta + \alpha + \epsilon\alpha$	$1 + \beta + \alpha + \epsilon\alpha$

5. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then

$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

   Answer:

   (a) Let the simple representations of G be $\sigma_1, \ldots, \sigma_r$; and let $\chi_i(g)$ be the character of σ_i.

   The simple characters $\chi_1, \ldots, \chi_r$ are linearly independent. For if say

$$\rho_1\chi_1(g) + \cdots + \rho_s\chi_s(g) = 0$$

   it follows from the formula for the intertwining number that for any representation α

$$\rho_1 I(\alpha, \sigma_1) + \cdots + \rho_r I(\alpha, \sigma_r) = 0.$$

   But on applying this with $\alpha = \sigma_i$ we deduce that $\rho_i = 0$ for each i.

   The characters are class functions:

$$\chi(gxg^{-1}) = \chi(x).$$

   The space of class functions has dimension s, the number of classes in G. It follows that $r \leq s$.

   To prove that $r = s$, it is sufficient to show that the characters span the space of class functions.

   Suppose $g \in G$ has order e. Let $[g]$ denote the class of g, and let $C = \langle g \rangle$ be the cyclic group generated by g.

   The group C has e 1-dimensional representations $\theta_1, \ldots, \theta_e$ given by

$$\theta_i : g \mapsto \omega^i,$$

   where $\omega = e^{2\pi i/e}$.

   Let

$$f(x) = \theta_0(x) + \omega^{-1}\theta_1(x) + \omega^{-2}\theta_2(x) + \cdots + \omega^{-e+1}\theta_{e-1}(x)$$

   Then

$$f(g^j) = \begin{cases} e & \text{if } j = 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Now let us "induce up" each of the characters θ_i from C to G. We have

$$\theta_i^G(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} \theta_i(y).$$

Let $F(x)$ be the same linear combination of the induced characters that $f(x)$ was of the θ_i. Then

$$F(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} f(y).$$

Since $f(y)$ vanishes away from g, we deduce that $F(x)$ vanishes off the class $[g]$, and is non-zero on that class:

$$F(x) \begin{cases} > 0 & \text{if } x \in [g], \\ = 0 & \text{if } x \notin [g]. \end{cases}$$

It follows that every class function on G can be expressed as a linear combination of characters, and therefore as a linear combination of simple characters. Hence the number of simple characters is at least as great as the number of classes.

We have shown therefore that the number of simple representations is equal to the number of classes.

(b) Consider the regular representation ρ of G. We have

$$\chi_\rho(g) = \begin{cases} |G| & \text{if } g = e, \\ 0 & \text{if } g \neq e. \end{cases}$$

Thus if α is any representation of G,

$$I(\rho, \alpha) = \chi_\alpha(e) = \dim \alpha.$$

Applying this to the simple representations $\alpha = \sigma_i$ we deduce that

$$\rho = (\dim \sigma_1)\sigma_1 + \cdots + (\dim \sigma_s)\sigma_s.$$

Taking dimensions on each side,

$$|G| = (\dim \sigma_1)^2 + \cdots + (\dim \sigma_s)^2.$$

(c) Since S_5 has 7 classes (corresponding to the types $1^5, 1^3 2, 1^2 3, 1 2^2, 14, 23, 5$) it has 7 simple representations.

We know that it has 2 representations of dimension 1, the trivial representation 1 and the parity representation ϵ.

Let ρ be the natural 5-dimensional representation of S_5 (by permutation of coordinates). Then it is easy to show directly that

$$\rho = 1 + \sigma$$

where σ is a simple 4-dimensional representation.

Now

$$\chi_\rho(1^3 2) = 3,$$

since a permutation of type $1^3 2$ leaves 3 elements fixed. It follows that

$$\chi_\sigma(1^3 2) = 2.$$

Hence

$$\epsilon\sigma \neq \sigma.$$

We now have 4 simple representations, of dimensions $1, 1, 4, 4$. *Let the other simple representations have dimensions* a, b, c. *Then*

$$a^2 + b^2 + c^2 = 120 - (1^2 + 1^2 + 4^2 + 4^2) = 86.$$

Since $m^2 \equiv 0$ *or* $1 \bmod 4$, *while* $86 \equiv 2 \bmod 4$, *it follows that 2 of* a, b, c *are odd and the other even.*

Let $t \in S_5$ *be a transposition. Then* $t^2 = 1$ *and so the eigenvalues of* t *in any representation* α *are* ± 1. *It follows that if* $\dim \alpha$ *is odd then* $\chi_\alpha(t) \neq 0$, *and therefore*

$$\epsilon\alpha \neq \alpha,$$

since the characters of these 2 representations differ at t. *In particular, representations of* S_5 *of odd dimension appear in pairs* $\sigma, \epsilon\sigma$.

Thus 2 of a, b, c *are equal odd numbers, say* $a = b$; *and*

$$2a^2 + c^2 = 86.$$

The only solution to this is $a = 5, c = 6$. *We conclude that the simple representations of* S_4 *have dimensions*

$$1, 1, 4, 4, 5, 5, 6.$$

6. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

   Answer:

   (a) We know that

   i. if $U \in \mathbf{SU}(2)$ *then* U *has eigenvalues*

   $$e^{\pm i\theta} \ (\theta \in \mathbb{R}).$$

   ii. if $X, Y \in \mathbf{GL}(n, k)$ *then*

   $$X \sim Y \implies X, Y \text{ have the same eigenvalues.}$$

   A fortiori, if $U \sim V \in \mathbf{SU}(2)$ *then* U, V *have the same eigenvalues.*

We shall show that the converse of the last result is also true, that is: $U \sim V$ in $\mathbf{SU}(2)$ if and only if U, V have the same eigenvalues $e^{\pm i\theta}$, This is equivalent to proving that

$$U \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

ie we can find $V \in \mathbf{SU}(2)$ such that

$$V^{-1}UV = U(\theta).$$

To see this, let v be an $e^{i\theta}$-eigenvalue of U. Normalise v, so that $v^*v = 1$; and let w be a unit vector orthogonal to v, ie $w^*w = 1$, $v^*w = 0$. Then the matrix

$$V = (vw) \in \mathbf{Mat}(2, \mathbb{C})$$

is unitary; and

$$V^{-1}UV = \begin{pmatrix} e^{i\theta} & x \\ 0 & e^{-i\theta} \end{pmatrix}$$

But in a unitary matrix, the squares of the absolute values of each row and column sum to 1. It follows that

$$|e^{i\theta}|^2 + |x|^2 = 1 \implies x = 0,$$

ie

$$V^{-1}UV = U(\theta).$$

We only know that $V \in \mathbf{U}(2)$, not that $V \in \mathbf{SU}(2)$. However

$$V \in \mathbf{U}(2) \implies |\det V| = 1 \implies \det V = e^{i\phi}.$$

Thus

$$V' = e^{-i\phi/2}V \in \mathbf{SU}(2)$$

and still

$$(V')^{-1}UV = U(\theta).$$

To summarise: Since $U(-\theta) \sim U(\theta)$ (by interchange of coordinates), we have shown that the conjugacy classes in $\mathbf{SU}(2)$ are

$$C(\theta) \quad (0 \leq \theta \leq \pi),$$

where

$$C(\theta) = \{U \in \mathbf{SU}(2) : U \text{ has eigenvalues } e^{\pm i\theta}\}$$

(b) *Suppose $m \in \mathbb{N}$, Let $V(m)$ denote the space of homogeneous polynomials $P(z, w)$ in z, w. Thus $V(m)$ is a vector space over $\mathbb{C}$ of dimension $m + 1$, with basis $z^m, z^{m-1}w, \ldots, w^m$.*
Suppose $U \in \mathbf{SU}(2)$. Then U acts on z, w by

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathbf{SU}(2)$ *on* $V(m)$*:*

$$P(z,w) \mapsto P(z',w').$$

We claim that the corresponding representation of $\mathbf{SU}(2)$ *— which we denote by* $D_{m/2}$ *— is simple, and that these are the only simple (finite-dimensional) representations of* $\mathbf{SU}(2)$ *over* $\mathbb{C}$*.*
To prove this, let

$$\mathbf{U}(1) \subset \mathbf{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$*. The action of* $\mathbf{SU}(2)$ *on* z, w *restricts to the action*

$$(z,w) \mapsto (e^{i\theta}z, e^{-i\theta}w)$$

of $\mathbf{U}(1)$*. Thus in the action of* $\mathbf{U}(1)$ *on* $V(m)$*,*

$$z^{m-r}w^r \mapsto e^{(m-2r)i\theta}z^{m-r}w^r,$$

It follows that the restriction of $D_{m/1}$ *to* $U(1)$ *is the representation*

$$D_{m/2}|\mathbf{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ *is the representation*

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathbf{U}(1)$*.*
In particular, the character of $D_{m/2}$ *is given by*

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2}i\theta + \cdots + e^{-mi\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$*.*
Now suppose $D_{m/2}$ *is* not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ *is semisimple, since* $\mathbf{SU}(2)$ *is compact.) Let a corresponding split of the representation space be*

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2}|\mathbf{U}(1)$ *are distinct, the expression of* $V(m)$ *as a direct sum of* $\mathbf{U}(1)$*-spaces,*

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1}w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique. It follows that W_1 *must be the direct sum of some of these spaces, and* W_2 *the direct sum of the others. In particular* $z^m \in W_1$ *or* $z^n \in W_2$*, say* $z^m \in W_1$*. Let*

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathbf{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z+w \\ -z+w \end{pmatrix}$$

under U*. Hence*

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r}w^r \rangle$*, it follows that*

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ *of* $\mathbf{SU}(2)$ *in* $V(m)$ *is simple.*
To see that every simple (finite-dimensional) representation of $\mathbf{SU}(2)$ *is of this form, suppose* α *is such a representation. Consider its restriction to* $\mathbf{U}(1)$*. Suppose*

$$\alpha|\mathbf{U}(1) = e_r E(r) + e_{r-1}E(r-1) + \cdots + e_{-r}E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α *has character*

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1}e^{(r-1)i\theta} + \cdots + e_{-r}e^{-ri\theta}$$

if U *has eigenvalues* $e^{\pm i\theta}$*.*
Since $U(-\theta) \sim U(\theta)$ *it follows that*

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots .$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ *with integer (possibly negative) coefficients:*

$$\chi(\theta) = a_0\chi_0(\theta) + a_{1/2}\chi_{1/2}(\theta) + \cdots + a_s\chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$*). Since* α *is simple,*

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j *is* ± 1 *and the rest are 0, ie*

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j*. But*

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

(c) *As we saw above, if U has eigenvalues $e^{\pm i\theta}$ then*

$$\chi_j(U) = e^{2ji\theta} + e^{2(j-1}i\theta + \cdots + e^{-2ji\theta}.$$

(d) *Finally, we show that*

$$D_j D_k = D_{j+k} + D_{j+k-1} + \cdots + D_{|j-k|}.$$

It is sufficient to prove the corresponding result for the characters

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k}(\theta) + \chi_{j+k-1}(\theta) + \cdots + \chi_{|j-k|}(\theta).$$

We may suppose that $j \geq k$. We prove the result by induction on k.

If $k = 0$ the result is trivial, since $\chi_0(\theta) = 1$. If $k = 1/2$ then

$$\begin{aligned}\chi_j(\theta)\chi_{1/2}(\theta) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)\\ &= \left(e^{(2j+1)i\theta} + e^{-(2j-1)i\theta}\right) + \left(e^{(2j-1)i\theta} + e^{-(2j+1)i\theta}\right)\\ &= \chi_{j+1/2}(\theta) + \chi_{j-1/2}(\theta),\end{aligned}$$

as required.

Suppose $k \geq 1$. Then

$$\chi_k(\theta) = \chi_{k-1}(\theta) + (e^{ki\theta} + e^{-ki\theta}).$$

Thus applying our inductive hypothesis,

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}).$$

But

$$\begin{aligned}\chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{ki\theta} + e^{-ki\theta}\right)\\ &= \chi_{j+k}(\theta) + \chi j - k(\theta),\end{aligned}$$

giving the required result

$$\begin{aligned}\chi_j(\theta)\chi_k(\theta) &= \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_{j+k}(\theta) + \chi j - k(\theta)\\ &= \chi_{j+k}(\theta) + \cdots + \chi_{j-k}.\end{aligned}$$

7. Define the *exponential* e^X of a square matrix X.

Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

Which of the above 5 matrices X are themselves expressible in the form $X = e^Y$ for some real matrix Y? (Justify your answers in all cases.)

Answer:

(a) The exponential of a square matrix X is defined by

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots .$$

[It is a straightforward matter to verify that this series always converges, using the fact that the matrix norm defined by

$$\|X\|^2 = \mathbf{tr}(X'X) = \sum_{i,j} |x_{ij}|$$

has the properties

i. $\|X + Y\| \leq \|X\| + \|Y\|$,
ii. $\|\lambda X\| = |\lambda|\|X\|$,
iii. $\|XY\| \leq \|X\|\|Y\|$.

However, I would not require this observation.]

i. If

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$e^X = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$$

ii. If

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = I,$$

and so

$$\begin{aligned} e^X &= (1 + 1/2! + 1/4! + \cdots)I + (1/1! + 1/3! + \cdots)X \\ &= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \end{aligned}$$

iii. If

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = -I,$$

and so

$$\begin{aligned} e^X &= (1 - 1/2! + 1/4! - \cdots)I + (1/1! - 1/3! + \cdots)X \\ &= \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \end{aligned}$$

iv. If

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + J,$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then (since I, J commute)

$$\begin{aligned} e^X &= e^I e^J \\ &= \begin{pmatrix} e\cos 1 & -e\sin 1 \\ e\sin 1 & e\cos 1 \end{pmatrix} \end{aligned}$$

(b) i. If

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\det X = -1,$$

and so $X \neq e^Y$ since

$$\det e^Y = e^{\mathbf{tr}Y} > 0.$$

ii. If

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then again

$$\det X < 0,$$

and so $X \neq e^Y$.

iii. By the isomorphims between the complex numbers $x + iy$ and the matrices

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

we see that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

corresponds to i.
Since

$$i = e^{\pi i/2}$$

we see that

$$J = e^Y$$

with

$$\begin{aligned} Y &= \frac{\pi}{2} J \\ &= \begin{pmatrix} 0 & -\pi/2 \\ \pi & 0 \end{pmatrix}. \end{aligned}$$

iv. Similarly

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

corresponds to the complex number $1 + i$. *Since*

$$\begin{aligned} 1 + i &= \sqrt{2}e^{\pi i/4} \\ &= e^{\log 2/2 + \pi i/4} \end{aligned}$$

$X = e^Y$ *with*

$$Y = \begin{pmatrix} \log 2/2 & -\pi/4 \\ \pi/4 & \log 2/2 \end{pmatrix}.$$

8. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathcal{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

   $$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}), \mathbf{GL}(n, \mathbb{C}), \mathbf{SL}(n, \mathbb{C})?$$

   Answer:

   (a) A linear group is a closed subgroup $G \subset \mathbf{GL}(n, \mathbb{R})$ *for some* n.

   (b) A Lie algebra is defined by giving

   i. a vector space L;

   ii. a binary operation on L, *ie a map*

   $$L \times L \to L : \ (X, Y) \mapsto [X, Y]$$

   satisfying the conditions

   i. The product $[X, Y]$ *is bilinear in* X, Y;

   ii. The product is skew-symmetric:

   $$[Y, X] = -[X, Y];$$

   iii. Jacobi's identity is satisfied:

   $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

   for all $X, Y, Z \in L$.

   (c) Suppose $G \subset \mathbf{GL}(n, \mathbb{R})$ *is a linear group. Then its Lie algebra* $L = \mathcal{L}G$ *is defined to be*

   $$L = \{X \in \mathbf{Mat}(n, \mathbb{R}) : e^{tX} \in G \ \forall t \in \mathbb{R}\}.$$

   (d) It follows at once from this definition that

   $$X \in L, \ \lambda \in R \implies \lambda X \in L.$$

   Thus to see that L *is a vector subspace of* $\mathbf{Mat}(n, \mathbb{R})$ *we must show that*

   $$X, Y \in L \implies X + Y \in L.$$

Now

$$\left(e^{X/n}e^{Y/n}\right)^n \mapsto e^{X+Y}$$

as $n \mapsto \infty$. (This can be seen by taking the logarithms of each side.) It follows that

$$X, Y \in L \implies e^{X+Y} \in G.$$

On replacing X, Y by tX, tY we see that

$$\begin{aligned} X, Y \in L &\implies e^{t(X+Y)} \in G \\ &\implies X + Y \in L. \end{aligned}$$

Similarly

$$\left(e^{X/n}e^{Y/n}e^{-X/n}e^{-Y/n}\right)^{n^2} \mapsto e^{[X,Y]},$$

as may be seen again on taking logarithms. It follows that

$$X, Y \in L \implies e^{[X,Y]} \in G.$$

Taking tX in place of X, this implies that

$$\begin{aligned} X, Y \in L &\implies e^{t[X,Y]} \in G \\ &\implies [X, Y] \in L. \end{aligned}$$

Thus L is a Lie algebra.

(e) The dimension of a linear group G is the dimension of the real vector space $\mathcal{L}G$:

$$\dim G = \dim_{\mathbb{R}} \mathcal{L}G.$$

(f) i. We have

$$\mathbf{o}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0\}$$

A skew symmetric matrix X is determined by giving the entries above the diagonal. This determines the entries below the diagonal; while those on the diagonal are 0. Thus

$$\dim O(n) = \dim o(n) = \frac{n(n-1)}{2}.$$

ii. We have

$$\mathbf{so}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0, \mathbf{tr}X = 0\} = \emptyset(n).$$

Since $X' + X = 0 \implies \mathbf{tr}X = 0$,

$$\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}.$$

iii. We have

$$\mathbf{u}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0\}$$

Again, the elements above the diagonal determine those below the diagonal; while those on the diagonal are purely imaginary. Thus

$$\begin{aligned} \dim \mathbf{U}(n) &= 2\frac{n(n-1)}{2} + n \\ &= n^2. \end{aligned}$$

iv. We have

$$\mathbf{su}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0,\ \mathbf{tr}X = 0\}$$

This gives one linear condition on the (purely imaginary) diagonal elements. Thus

$$\dim \mathbf{SU}(n) = \dim \mathbf{U}(n) - 1 = n^2 - 1.$$

v. We have

$$\mathbf{gl}(n, \mathbb{R}) = \mathbf{Mat}(n, \mathbb{R}).$$

Thus

$$\dim \mathbf{GL}(n, \mathbb{R}) = n^2.$$

vi. We have

$$\mathbf{sl}(n, \mathbb{R}) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : \mathbf{tr}X = 0\}.$$

This imposes one linear condition on X. Thus

$$\dim \mathbf{SL}(n, \mathbb{R}) = \dim \mathbf{GL}(n, \mathbb{R}) - 1 = n^2 - 1.$$

vii. We have

$$\mathbf{gl}(n, \mathbb{C}) = \mathbf{Mat}(n, \mathbb{C}).$$

Each of the n^2 complex entries takes 2 real values. Thus

$$\dim \mathbf{GL}(n, \mathbb{C}) = 2n^2.$$

viii. We have

$$\mathbf{sl}(n, \mathbb{C}) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : \mathbf{tr}X = 0\}.$$

This imposes one complex linear condition on X, or 2 real linear conditions. Thus

$$\dim \mathbf{SL}(n, \mathbb{C}) = \dim \mathbf{GL}(n, \mathbb{C}) - 1 = 2n^2 - 2.$$

9. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

   Answer:

(a) We have

$$\begin{aligned}\mathbf{u}(2) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) :\ e^{tX} \in \mathbf{U}(2)\ \forall t \in \mathbb{R}\}\\ &= \{X : (e^{tX})^* e^{tX} = I\ \forall t\}\\ &= \{X : e^{tX^*} = e^{-tX}\ \forall t\}\\ &= \{X : X^* = -X\}\\ &= \{X : X^* + X = 0\},\end{aligned}$$

while

$$\begin{aligned}\mathbf{sl}(2,\mathbb{C}) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) :\ e^{tX} \in \mathbf{SL}(2,\mathbb{C})\ \forall t \in \mathbb{R}\}\\ &= \{X : \det e^{tX} = 1\ \forall t\}\\ &= \{X : e^{t\mathbf{tr}X} = 1\ \forall t\}\\ &= \{X : \mathbf{tr}X = 0\}.\end{aligned}$$

Since

$$\mathbf{SU}(2) = \mathbf{U}(2) \cap \mathbf{SL}(2,\mathbb{C})$$

it follows that

$$\begin{aligned}\mathbf{su}(2) &= \mathbf{u}(2) \cap \mathbf{sl}(2,\mathbb{C})\\ &= \{X : X^* + X = 0,\ \mathbf{tr}X = 0\}.\end{aligned}$$

The 3 matrices

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{su}(2)$.
We have

$$\begin{aligned}[e,f] = ef - fe = -2g,\\ [e,g] = eg - ge = 2f,\\ [f,g] = fg - gf = -2e\end{aligned}$$

Thus

$$\mathbf{su}(2) = \langle e, f, g : [e,f] = -2g, [e,g] = 2f, [f,g] = -2e\rangle.$$

(b) We have

$$\begin{aligned}\mathbf{o}(3) &= \{X \in \mathbf{Mat}(3,\mathbb{R}) :\ e^{tX} \in \mathbf{O}(3)\ \forall t \in \mathbb{R}\}\\ &= \{X : (e^{tX})' e^{tX} = I\ \forall t\}\\ &= \{X : e^{tX'} = e^{-tX}\ \forall t\}\\ &= \{X : X' = -X\}\\ &= \{X : X' + X = 0\},\end{aligned}$$

while

$$\begin{aligned}\mathbf{sl}(3,\mathbb{R}) &= \{X \in \mathbf{Mat}(3,\mathbb{R}) :\ e^{tX} \in \mathbf{SL}(3,\mathbb{R})\ \forall t \in \mathbb{R}\}\\ &= \{X : \det e^{tX} = 1\ \forall t\}\\ &= \{X : e^{t\mathbf{tr}X} = 1\ \forall t\}\\ &= \{X : \mathbf{tr}X = 0\}.\end{aligned}$$

Since

$$\mathbf{SO}(3) = \mathbf{O}(3) \cap \mathbf{SL}(3,\mathbb{R})$$

it follows that

$$\begin{aligned}\mathbf{so}(3) &= \mathbf{o}(3) \cap \mathbf{sl}(3,\mathbb{R})\\ &= \{X : X' + X = 0,\ \mathbf{tr}X = 0\}\\ &= \{X : X' + X = 0\}\end{aligned}$$

since a skew-symmetric matrix necessarily has trace 0.
The 3 matrices

$$U = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\ V = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\ W = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

form a basis for the vector space $\mathbf{so}(3)$.
We have

$$[V, W] = U;$$

and so by cyclic permutation of indices (or coordinates)

$$[W, U] = V,\ [U, V] = W.$$

Thus

$$\mathbf{so}(3) = \langle U, V, W : [U, V] = W, [U, W] = -V, [V, W] = U\rangle.$$

(c) The Lie algebras $su(2)$ *and* $so(3)$ *are isomorphic under the correspondence*

$$e \leftrightarrow -2U,\ f \leftrightarrow -2V,\ g \leftrightarrow -2W.$$

(d) However, the groups $\mathbf{SU}(2), \mathbf{SO}(3)$ *are not isomorphic, since*

$$Z\mathbf{SU}(2) = \{\pm I\}\ \textit{while}\ Z\mathbf{SO}(3) = \{I\}.$$

10. Define a *representation* of a Lie algebra $\mathcal{L}$. What is meant by saying that such a representation is (a) *simple*, (b) *semisimple*?

    Determine the Lie algebra of $\mathbf{SL}(2,\mathbb{R})$, and find all the simple representations of this algebra.

    Show that every representation of the group $\mathbf{SL}(2,\mathbb{R})$ is semisimple, stating carefully but without proof any results you need.

    Answer:

(a) *Suppose L is a real Lie algebra. A representation of L in the complex vector space V is defined by giving a map*

$$L \times V \to V : \ (X, v) \mapsto Xv$$

which is bilinear over $\mathbb{R}$ and which satisfies the condition

$$[X, Y]v = X(Yv) - Y(Xv)$$

for all $X, Y \in L, v \in V$.

A representation of L in V is thus the same as a representation of the complexification $L_{\mathbb{C}}$ of L in V.

(b) *The representation is said to be simple if no proper subspace $U \subset V$ is stable under L.*

It is said to be semisimple if V is a direct sum of simple subspaces.

(c) *We have*

$$\begin{aligned}\mathbf{sl}(2, \mathbb{R}) &= \{X \in \mathbf{Mat}(2, \mathbb{R}) : \ e^{tX} \in \mathbf{SL}(2, \mathbb{R}) \forall t \in \mathbb{R}\} \\ &= \{X : \det e^{tX} = 1 \forall t\} \\ &= \{X : e^{t\mathbf{tr}X} = 1 \forall t\} \\ &= \{X : \mathbf{tr}X = 0\}.\end{aligned}$$

The 3 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{sl}(2, \mathbb{R})$.

We have

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = H.$$

Thus

$$\mathbf{sl}(2, \mathbb{R}) = \langle H, E, F : [H, E] = 2E, [H, F] = -2F, [V, W] = H \rangle.$$

(d) *Suppose we have a simple representation of $\mathbf{sl}(2, \mathbb{R})$ on V. Let v be an eigenvector of H with eigenvalue λ:*

$$Hv = \lambda v.$$

Then

$$[H, E]v = 2Ev,$$

ie

$$HEv - EHv = 2Ev.$$

In other words, since $Hv = \lambda v$,

$$H(Ev) = (\lambda + 2)Ev,$$

that is, Ev is an eigenvector of H with eigenvalue $\lambda + 2$.
By the same argument $E^2v, E^3v, \ldots$ are all eigenvectors of H with eigenvalues $\lambda + 4, \lambda + 6, \ldots$, at least until they vanish.
This must happen at some point, since V is finite-dimensional; say

$$E^{r+1}v = 0, E^r v \neq 0.$$

Similarly we find that

$$Fv, F^2v, \ldots$$

are also eigenvectors of H (until they vanish) with eigenvalues $\lambda - 2, \lambda - 4, \ldots$. Again we must have

$$F^{s+1}v = 0, F^s v \neq 0$$

for some s.
Now let us write e_0 for $F^s v$, so that

$$Fe_0 = 0;$$

and let $\mu = \lambda - 2s$, so that e_0 is a μ-eigenvector of H:

$$He_0 = \mu e_0.$$

Set

$$e_i = E^i e_0,$$

so that

$$e_{i+1} = Ee_i$$

for $i \geq 0$. Thus e_i is a $(\mu + 2i)$-eigenvector of H; and if $m = r + s$ then

$$e_m \neq 0,\ e_{m+1} = Ee_m = 0.$$

By the argument above, Fe_i is an eigenvector of H with the same eigenvalue, $\mu + 2(i-1)$, as e_{i-1}. We shall show by induction on i that Fe_i is a scalar multiple of e_{i-1}, say

$$Fe_i = \rho_i e_{i-1}.$$

The result holds for $i = 0$ with $\rho_0 = 0$. Suppose it holds for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned} Fe_{m+1} &= FEe_m \\ &= (EF - [E, F])e_m \\ &= \rho_m Ee_{m-1} - He_m \\ &= (\rho_m - \mu - 2m)e_m. \end{aligned}$$

This proves the result, and also shows that

$$\rho_{i+1} = \rho_i - \mu - 2i$$

for each i. It follows that

$$\rho_i = -i\mu - i(i-1).$$

We must have $\rho_{m+1} = 0$. Hence

$$\mu = m.$$

We conclude that the subspace

$$\langle e_0, \ldots, e_m \rangle$$

is stable under $\mathbf{sl}(2,\mathbb{R})$, and so must be the whole of V.

Thus we have shown that there is at most 1 simple representation of each dimension n, and we have determined this explicitly, if it exists. In fact it is a straightforward matter to verify that the above actions of H, E, F on $\langle e_0, \ldots, e_m \rangle$ do indeed define a representation of $\mathbf{sl}(2,\mathbb{R})$; so that this Lie algebra has exactly 1 simple representation of each dimension.

(e)

i. *The representations over $\mathbb{C}$ of a real Lie algebra L correspond to the representations of its complexification $\mathbb{C}L$.*

ii. *Thus if two real Lie algebras have isomorphic complexifications then there is a one-one correspondence between their representations, with simple or semisimple representations of one corresponding to simple or semisimple representations of the other.*

iii. *The Lie algebras $\mathbf{sl}(n,\mathbb{R})$ and $\mathbf{su}(n)$ have the same complexification:*

$$\mathbb{C}\mathbf{sl}(n,\mathbb{R}) = \mathbf{sl}(c,\mathbb{C}) = \mathbb{C}\mathbf{su}(n).$$

iv. *The representations of $\mathbf{su}(2)$ can all be lifted to representations of $\mathbf{SU}(2)$, since $\mathbf{SU}(2) \cong S^3$ is simply-connected.*

v. *The representations of $\mathbf{SU}(2)$ are all semisimple, since $\mathbf{SU}(2)$ is compact.*

vi. *Hence the representations of $\mathbf{su}(2$ are all semisimple, and so the representations of $\mathbf{sl}(2,\mathbb{R})$ are all semisimple.*