Exercises 1

All representations are over $\mathbb{C}$, unless the contrary is stated.

In Exercises 01–11 determine all 1-dimensional representations of the given group.

1 $*$ C_2	2 $**$ C_3	3 $**$ C_n	4 $**$ D_2	5 $**$ D_3
6 $***$ D_n	7 $***$ Q_8	8 $***$ A_4	9 $****$ A_n	10 $**$ $\mathbb{Z}$
11 $****$ $D_\infty = \langle r, s : s^2 = 1, rsr = s \rangle$				

Suppose G is a group; and suppose $g, h \in G$. The element $[g,h] = ghg^{-1}h^{-1}$ is called the *commutator* of g and h. The subgroup $G' \equiv [G,G]$ is generated by all commutators in G is called the commutator subgroup, or *derived group* of G.

12 $***$ Show that G' lies in the kernel of any 1-dimensional representation ρ of G, ie $\rho(g)$ acts trivially if $g \in G'$.

13 $***$ Show that G' is a normal subgroup of G, and that G/G' is abelian. Show moreover that if K is a normal subgroup of G then G/K is abelian if and only if $G' \subset K$. [In other words, G' is the smallest normal subgroup such that G/G' is abelian.)

14 $**$ Show that the 1-dimensional representations of G form an abelian group G^* under multiplication. [Nb: this notation G^* is normally only used when G is abelian.]

15 $**$ Show that $C_n^* \cong C_n$.

16 $***$ Show that for any 2 groups G, H

$$(G \times H)^* = G^* \times H^*.$$

17 $****$ By using the Structure Theorem on Finite Abelian Groups (stating that each such group is expressible as a product of cyclic groups) or otherwise, show that

$$A^* \cong A$$

for any finite abelian group A.

18 $**$ Suppose $\Theta : G \to H$ is a homomorphism of groups. Then each representation α of H defines a representation $\Theta\alpha$ of G.

19 $***$ Show that the 1-dimensional representations of G and of G/G' are in one-one correspondence.

In Exercises 20–24 determine the derived group G' of the given group G.

20 $***$ C_n	21 $****$ D_n	22 $**$ $\mathbb{Z}$	23 $****$ D_∞
24 $***$ Q_8	25 $***$ S_n	26 $***$ A_4	27 $****$ A_n

Exercises 2

All representations are over $\mathbb{C}$, unless the contrary is stated.

In Exercises 01–15 determine all 2-dimensional representations (up to equivalence) of the given group.

1 ** C_2	2 ** C_3	3 ** C_n	4 *** D_2	5 *** D_4
6 *** D_5	7 **** D_n	8 *** S_3	9 **** S_4	10 ***** S_n
11 **** A_4	12 ***** A_n	13 *** Q_8	14 ** $\mathbb{Z}$	15 **** D_∞

16 *** Show that a real matrix $A \in \mathbf{Mat}(n, \mathbb{R})$ is diagonalisable over $\mathbb{R}$ if and only if its minimal polynomial has distinct roots, all of which are real.

17 *** Show that a rational matrix $A \in \mathbf{Mat}(n, \mathbb{Q})$ is diagonalisable over $\mathbb{Q}$ if and only if its minimal polynomial has distinct roots, all of which are rational.

18 **** If 2 real matrices $A, B \in \mathbf{Mat}(n, \mathbb{R})$ are similar over $\mathbb{C}$, are they necessarily similar over $\mathbb{R}$, ie can we find a matrix $P \in \mathbf{GL}(n, \mathbb{R})$ such that $B = PAP^{-1}$?

19 **** If 2 rational matrices $A, B \in \mathbf{Mat}(n, \mathbb{Q})$ are similar over $\mathbb{C}$, are they necessarily similar over $\mathbb{Q}$?

20 ***** If 2 integral matrices $A, B \in \mathbf{Mat}(n, \mathbb{Z})$ are similar over $\mathbb{C}$, are they necessarily similar over $\mathbb{Z}$, ie can we find an integral matrix $P \in \mathbf{GL}(n, \mathbb{Z})$ with integral inverse, such that $B = PAP^{-1}$?

The matrix $A \in \mathbf{Mat}(n, k)$ is said to be *semisimple* if its minimal polynomial has distinct roots. It is said to be *nilpotent* if $A^r = 0$ for some $r > 0$.

21 *** Show that a matrix $A \in \mathbf{Mat}(n, k)$ cannot be both semisimple and nilpotent, unless $A = 0$.

22 *** Show that a polynomial $p(x)$ has distinct roots if and only if

$$\gcd\,(p(x), p'(x)) = 1.$$

23 **** Show that every matrix $A \in \mathbf{Mat}(n, \mathbb{C})$ is uniquely expressible in the form

$$A = S + N,$$

where S is semisimple, N is nilpotent, and

$$SN = NS.$$

(We call S and N the semisimple and nilpotent parts of A.)

24 **** Show that S and N are expressible as polynomials in A.

25 **** Suppose the matrix $B \in \mathbf{Mat}(n, \mathbb{C})$ commutes with all matrices that commute with A, ie

$$AX = XA \Longrightarrow BX = XB.$$

Show that B is expressible as a polynomial in A.

Exercises 3

In Exercises 01–10 determine all simple representations of the given group over $\mathbb{C}$.

1 $**$ C_2	2 $**$ C_3	3 $**$ C_n	4 $***$ D_2	5 $***$ D_4
6 $***$ D_5	7 $****$ D_n	8 $***$ S_3	9 $****$ A_4	10 $*****$ Q_8

In Exercises 11–20 determine all simple representations of the given group over $\mathbb{R}$.

11 $**$ C_2	12 $***$ C_3	13 $***$ C_n	14 $***$ D_2	15 $****$ D_4
16 $****$ D_5	17 $****$ D_n	18 $****$ S_3	19 $*****$ A_4	20 $*****$ Q_8

In Exercises 21–25 determine all simple representations of the given group over the rationals$\mathbb{Q}$.

21 $****$ C_n	22 $****$ D_n	23 $****$ S_3	24 $***$ Q_8	25 $*****$ A_4