Course 424 — Group Representations

Sample Exam

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Attempt 7 questions. (If you attempt more, only the best 7 will be counted.) All questions carry the same number of marks.
In this paper representation *means "finite-dimensional representation over $\mathbb{C}$".*

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Find all representations of S_3 of degree 2 (up to equivalence).

   What is meant by saying that a representation α is *simple*? Find all simple representations of D_4 from first principles.

   Answer:

   (a) *A representation α of a group G in a vector space V is a homomorphism*
   $$\alpha : G \to \mathbf{GL}(V).$$

   (b) *The representations α, β of G in U, V respectively are said to be equivalent if U, V are isomorphic as G-spaces, that is, there exists an invertible map*
   $$t : U \to V$$
   which preserves the action of G, ie
   $$t(gu) = g(tu)$$
   for all $g \in G, u \in U$.

   (c) *Suppose α is a representation of*
   $$S_3 = \langle s, t : s^3 = t^2 = 1, st = ts^2 \rangle$$
   (taking $s = (abc), t = (ab)$) in the 2-dimensional vector space V.

Let e be an eigenvector of s with eigenvalue λ:

$$se = \lambda e.$$

Since $s^3 = 1$, $\lambda \in \{1, \omega, \omega^2\}$ (where $\omega = e^{2\pi/3}$).
Let $f = te$. Then

$$sf = ste = ts^2 e = \lambda^2 te = \lambda^2 f.$$

In other words, f is also an eigenvector of s, but with eigenvalue λ^2.
If $\lambda^2 \neq \lambda$, ie $\lambda \neq 1$, then e, f are eigenvectors of s with different eigenvalues, and so are necessarily linearly independent. Thus in this case e, f is a basis for V, and the representation is completely determined. If $\lambda = \omega$ then

$$s \to \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}, \quad t \to \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If $\lambda = \omega^2$ then f has eigenvalue $(\omega^2)^2 = \omega$, and we get the same representation with e, f interchanged. Thus there is just one representation in this case.
We must fall into this case unless both *the eigenvalues of s are 1. In the latter case, since $\alpha(g)$ is always semisimple (diagonalisable), we must have $s \to I$ with respect to any basis.*
But then we can diagonalise t without effecting s. Since $t^2 = 1$,

$$t \to \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

Two of these representations are equivalent; so we get just 3 representations (in effect $1+1, 1+\epsilon, \epsilon+\epsilon$) in this case:

$$t \to \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus S_3 has just 4 representations of degree 2.

(d) *The representation α of G in V is said to be simple if no subspace $U \subset V$ is stable under G except for $U = 0, V$. (The subspace U is said to be stable under G if*

$$g \in G, u \in U \implies gu \in U.)$$

(e) *We have*

$$D_4 = \langle t, s : s^4 = t^2 = 1, st = ts^3 \rangle.$$

Let us first suppose α is a 1-dimensional representations of D_4. ie a homomorphism

$$\alpha : D_4 \to \mathbb{C}^*.$$

Suppose

$$\alpha(s) = \lambda, \ \alpha(t) = \mu.$$

Then
$$\lambda^4 = \mu^2 = 1, \ \lambda\mu = \mu\lambda^3.$$
The last relation gives
$$\lambda^2 = 1.$$
Thus there are just 4 1-dimensional representations given by
$$s \mapsto \pm 1, t \mapsto \pm 1.$$
Now suppose α *is a simple representation of* G *in the vector space* V *over* $\mathbb{C}$*, where* $\dim V \geq 2$*. Let* $e \in V$ *be an eigenvector of* s*:*
$$se = \lambda e;$$
and let
$$f = te.$$
Then
$$sf = ste = ts^3 e = \lambda^3 te = \lambda^3 f,$$
ie f *is a* λ^3*-eigenvector of* s*.*
It follows that the subspace
$$\langle e, f \rangle \subset V$$
is stable under D_4*, since*
$$se = \lambda e, \ sf = \lambda^3 f, \ te = f, \ tf = t^2 e = e.$$
Since V *by definition is simple, it follows that*
$$V = \langle e, f \rangle.$$
In particular, $\dim \alpha = 2$*, and* e, f *form a basis for* V*.*
Since $s^4 = 1$ *we have* $\lambda^4 = 1$*, ie* $\lambda \in \{\pm 1, \pm i\}$*.*
If $\lambda = 1$ *then* s *would have eigenvalues* $1, 1$ *(since* $1^3 = 1$*). But we know that* s *(ie* $\alpha(s)$*) is diagonalisable. It follows that* $s = I$*.*
Similarly if $\lambda = -1$ *then* s *has eigenvalues* $-1, -1$ *and so* $s = -I$*.*
In either of these cases s *will always be diagonal. Since we can always diagonalise* t*, we can diagonalise* s, t *simultaneously. But in that case the representation would not be simple; for if* e *is a common eigenvector of* s, t *then the 1-dimensional space* $\langle e \rangle$ *is stable under* D_4*.*
Thus we are left with the cases $\lambda = \pm i$*. If* $\lambda = -i$ *then on swapping* e *and* f *we would have* $\lambda = i$*. So we have only 1 2-dimensional representation (up to equivalence):*
$$s \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
In conclusion, D_4 *has just 5 simple representations: 4 of dimension 1, and 1 of dimension 2.*

2. What is meant by saying that a representation α is *semisimple*?

   State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

   Prove that every representation of a compact group is semisimple.

   Answer:

   (a) *The representation α of G in V is said to be semisimple if it can be expressed as a sum of simple representations:*

   $$\alpha = \sigma_1 + \cdots + \sigma_m.$$

   This is equivalent to the condition that each stable subspace $U \subset V$ has a stable complement W:

   $$V = U \oplus W.$$

   (b) *Haar's theorem states that there exists an invariant measure on a compact group G, unique up to a scalar multiple.*
   A measure *μ on X is a continuous linear functional*

   $$\mu : C(X) \to \mathbb{C},$$

   where $C(X) = C(X, \mathbb{R})$ is the space of real-valued continuous functions on X with norm $\|f\| = \sup |f(x)|$.
   The compact group G acts on $C(G)$ by

   $$(gf)(x) = f(g^{-1}x).$$

   The measure μ is said to be invariant under G if

   $$\mu(gf)\mu(f)$$

   for all $g \in G,\ f \in C(G)$.

   (c) *We can prove the existence of such an invariant measure in the following way.*
   By an average *F of $f \in C(G)$ we mean a function of the form*

   $$F = \lambda_1 g_1 f + \lambda_2 g_2 f + \cdots + \lambda_r g_r f,$$

   ie

   $$F(x) = \lambda_1 f(g_1^{-1}x) + \lambda_2 f(g_2^{-1}x) + \cdots + \lambda_r f(g_r^{-1}x).$$

   where $0 \le \lambda_i \le 1$, $\sum \lambda_i = 1$ and $g_1, g_2, \ldots, g_r \in G$.
   If F is an average of f then

   i. $\inf f \le \inf F \le \sup F \le sup f$;
   ii. If μ is an invariant measure then $\mu(F) = \mu(f)$;
   iii. An average of F is an average of f.

Let us set

$$\operatorname{var}(f) = \sup f - \inf f$$

so that

$$\operatorname{var}(F) \le \operatorname{var}(f)$$

for any average F of f.

We shall establish a sequence of averages $F_0 = f, F_1, F_2, \ldots$ (each an average of its predecessor) such that $\operatorname{var}(F_i) \to 0$*. It will follow that*

$$F_i \to c \in \mathbb{R},$$

ie $F_i(g) \to c$ for each $g \in G$.

Suppose $f \in C(G)$. It is not hard to find an average F of f with $\operatorname{var}(F) < \operatorname{var}(f)$*. Let*

$$V = \{g \in G : f(g) < \frac{1}{2}(\sup f + \inf f),$$

ie V is the set of points where f is 'below average'. Since G is compact, we can find $g_1, \ldots, g_r$ such that

$$G = g_1V \cup \cdots \cup g_rV.$$

Consider the average

$$F = \frac{1}{r}\left(g_1f + \cdots + g_rf\right).$$

Suppose $x \in G$. Then $x \in g_iV$ for some i, ie

$$g_i^{-1}x \in V.$$

Hence

$$(g_if)(x) = f(g_i^{-1}x) < \frac{1}{2}(\sup f + \inf f),$$

and so

$$\begin{aligned} F(x) &< \frac{r-1}{r}\sup f + \frac{1}{2r}(\sup f + \inf f) \\ &= \sup f - \frac{1}{2r}\sup f - \inf f. \end{aligned}$$

Hence $\sup F <$ *supf and so*

$$\operatorname{var}(F) < \operatorname{var}(f).$$

This allows us to construct a sequence of averages $F_0 = f, F_1, F_2, \ldots$ such that

$$\operatorname{var}(f) = \operatorname{var}(F_0) > \operatorname{var}(F_1) > \operatorname{var}(F_2) > \cdots.$$

But that is not sufficient to show that $\operatorname{var}(F_i) \to 0$*. For that we must use the fact that any $f \in C(G)$ is* uniformly *continuous.*

[I would accept this last remark as sufficient in the exam, and would not insist on the detailed argument that follows.]

In other words, given $\epsilon > 0$ we can find an open set $U \ni e$ such that

$$x^{-1}y \in U \implies |f(x) - f(y)| < \epsilon.$$

Since

$$(g^{-1}x)^{-1}(g^{-1}y) = x^{-1}y,$$

the same result also holds for the function gf. Hence the result holds for any average F of f.

Let V be an open neighbourhood of e such that

$$VV \subset U, \quad V^{-1} = V.$$

(If V satisfies the first condition, then $V \cap V^{-1}$ satisfies both conditions.) Then

$$xV \cup yV \neq \emptyset \implies |f(x) - f(y)| < \epsilon.$$

For if $xv = yv'$ then

$$x^{-1}y = vv'^{-1} \in U.$$

Since G is compact we can find $g_1, \ldots, g_r$ such that

$$G = g_1V \cup \cdots \cup g_rV.$$

Suppose f attains its minimum $\inf f$ at $x_0 \in g_iV$; and suppose $x \in g_jV$. Then

$$g_i^{-1}x_0,\ g_j^{-1}x \in V.$$

Hence

$$\left(g_j^{-1}x\right)^{-1}\left(g_i^{-1}x_0\right) = \left(g_ig_j^{-1}x\right)^{-1}x_0 \in U,$$

and so

$$|f(g_ig_j^{-1}x) - f(x_0)| < \epsilon.$$

In particular,

$$(g_jg_i^{-1}f)(x) < \inf f + \epsilon.$$

Let F be the average

$$F = \frac{1}{r^2}\sum_{i,j} g_jg_i^{-1}f.$$

Then

$$\sup F < \frac{r^2-1}{r^2}\sup f + \frac{1}{r^2}(\inf f + \epsilon),$$

and so

$$\begin{aligned}\operatorname{var}(F) &< \frac{r^2-1}{r^2}\operatorname{var}(f) + \frac{1}{r^2}\epsilon \\ &< \frac{r^2-1/2}{r^2}\operatorname{var}(f),\end{aligned}$$

if $\epsilon < \mathrm{var}(f)/2$.
Moreover this result also holds for any average of f in place of f. It follows that a succession of averages of this kind

$$F_0 = f, F_1, \ldots, F_s$$

will bring us to

$$\mathrm{var}(F)_s < \frac{1}{2}\,\mathrm{var}(f).$$

Now repeating the same argument with F_s, and so on, we will obtain a sequence of successive averages $F_0 = f, F_1, \ldots$ with

$$\mathrm{var}(F)_i \downarrow 0.$$

It follows that

$$F_i \to c$$

(the constant function with value c).
It remains to show that this limit value c is unique. For this we introduce right averages

$$H(x) = \sum_j \mu_j f(xh_j)$$

where $0 \le \mu_j \le 1$, $\sum \mu_j = 1$. (Note that a right average of f is in effect a left average of $\tilde{f}$, where $\tilde{f}(x) = f(x^{-1})$. In particular the results we have established for left averages will hold equally well for right averages.)
Given a left average and a right average of f, say

$$F(x) = \sum \lambda_i f(g_i^{-1}x), \quad H(x) = \sum \mu_j f(xh_j),$$

we can form the joint average

$$J(x) = \sum_{i,j} \lambda_i \mu_j f(g_i^{-1}xh_j).$$

It is easy to see that

$$\begin{aligned} \inf F \le \inf J \le \sup J \le \sup H, \\ \sup F \ge \sup J \ge \inf J \ge \inf H. \end{aligned}$$

But if now $H_0 = f, H_1, \ldots$ is a succession of right averages with $H_i \to d$ then it follows that

$$c = d.$$

In particular, any two convergent sequences of successive left averages must tend to the same limit. We can therefore set

$$\mu(f) = c.$$

Thus $\mu(f)$ is well-defined; and it is invariant since f and gf have the same set of averages. Finally, if $f = 1$ then $\mathrm{var}(f) = 0$, and $f, f, f, \ldots$ converges to 1, so that

$$\mu(1) = 1.$$

The invariant measure on G is unique up to a scalar multiple. In other words, it is unique if we normalise the measure by specifying that

$$\mu(1) = 1$$

(where 1 on the left denotes the constant function 1).

(d) *Suppose α is a representation of the finite group G in the vector space V. Let*

$$P(u, v)$$

be a positive-definite hermitian form on V. Define the hermitian form Q on V by

$$Q(u, v) = \frac{1}{\|G\|} \sum_{g \in G} H(gu, gv).$$

Then Q is positive-definite (as a sum of positive-definite forms). Moreover Q is invariant under G, ie

$$Q(gu, gv) = Q(u, v)$$

for all $g \in G, u, v \in V$. For

$$\begin{aligned} Q(hu, hv) &= \frac{1}{\|G\|} \sum_{g \in G} H(ghu, ghv) \\ &= \frac{1}{|G|} \sum_{g \in G} H(gu, gv) \\ &= Q(u, v), \end{aligned}$$

since gh runs over G as g does.

Now suppose U is a stable subspace of V. Then

$$U^{\perp} = \{v \in V : Q(u, v) = 0 \forall u \in U\}$$

is a stable complement to U.

Thus every stable subspace has a stable complement, ie the representation is semisimple.

3. Define the *character* χ_α of a representation α, and show that it is a class function (ie it is constant on conjugacy classes).

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β of a group G, and show that if G is compact then

$$I(\alpha, \beta) = \int_G \overline{\chi_\alpha(g)} \chi_\beta(g) \, dg.$$

   Prove that a representation α is simple if and only if $I(\alpha, \alpha) = 1$.

4. Draw up the character table for S_4.

Determine also the representation-ring for this group, ie express the product $\alpha\beta$ of each pair of simple representation as a sum of simple representations.

Answer:

(a) *S_4 has 5 classes, corresponding to the types $1^4, 1^2 2, 13, 2^2, 4$. Thus S_4 has 5 simple representations.*

Each symmetric group S_n (for $n \geq 2$) has just 2 1-dimensional representations, the trivial representation 1 and the parity representation ϵ.

Let $S_4 = \mathbf{Perm}(X)$, where $X = \{a, b, c, d\}$. The action of S_4 on X defines a 4-dimensional representation ρ of S_4, with character

$$\chi(g) = |\{x \in X : gx = x\}|$$

In other words $\chi(g)$ is just the number of 1-cycles in g.

So now we can start our character table (where the second line gives the number of elements in the class):

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
ρ	4	2	1	0	0

Now

$$I(\rho, \rho) = \frac{1}{24}(1 \cdot 16 + 6 \cdot 4 + 8 \cdot 1) = 2.$$

It follows that ρ has just 2 simple parts. Since

$$I(1, \rho) = \frac{1}{24}(1 \cdot 4 + 6 \cdot 2 + 8 \cdot 1) = 1,$$

It follows that

$$\rho = 1 + \alpha,$$

where α is a simple 3-dimensional representation, with character given by

$$\chi(g) = \chi_\rho(g) - 1.$$

The representation $\epsilon\alpha$ is also simple, and is not equal to α since it has a different character. So now we have 4 simple characters of S_4, as follows:

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1

To find the 5th simple representation, we can consider α^2. This has character

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
α^2	9	1	0	1	1

We have

$$
\begin{aligned}
I(1, \alpha^2) &= \frac{1}{24}(9 + 6 + 3 + 6) = 1, \\
I(\epsilon, \alpha^2) &= \frac{1}{24}(9 - 6 + 3 - 6) = 0, \\
I(\alpha, \alpha^2) &= \frac{1}{24}(27 + 6 - 3 - 6) = 1, \\
I(\epsilon\alpha, \alpha^2) &= \frac{1}{24}(27 - 6 - 3 + 6) = 1. I(\alpha^2, \alpha^2) \quad = \frac{1}{24}(81 + 6 + 3 + 6) = 4,
\end{aligned}
$$

It follows that α^2 has 4 simple parts, so that

$$\alpha^2 = 1 + \alpha + \epsilon\alpha + \beta,$$

where β is the 5th simple representation, with character given by

$$\chi_\beta(g) = \chi_\alpha(g)^2 - 1 - \chi_\alpha(g) - \epsilon(g)\chi_\alpha(g).$$

This allows us to complete the character table:

	1^4	$1^2 2$	13	2^2	4
	(1)	(6)	(8)	(3)	(6)
1	1	1	1	1	1
ϵ	1	-1	1	1	-1
α	3	1	0	-1	-1
$\epsilon\alpha$	3	-1	0	-1	1
β	2	0	-1	2	0

(b) *We already know how to express α^2 in terms of the 5 simple representations. Evidently $\epsilon\beta = \beta$ since there is only 1 simple representation of dimension 2. The character of $\alpha\beta$ is given by*

	1^4	$1^2 2$	13	2^2	4
$\alpha\beta$	6	0	0	-2	0

We have

$$I(\alpha\beta, \alpha\beta) = \frac{1}{24}(36 + 12) = 2.$$

Thus $\alpha\beta$ has just 2 simple parts. These must be α and $\epsilon\alpha$ to give dimension 6:

$$\alpha\beta = \alpha + \epsilon\alpha.$$

Also we have

$$I(\beta^2, \beta^2) = \frac{1}{24}(16 + 8 + 48) = 3.$$

Thus β has 3 simple parts. So by dimension, we must have

$$\beta^2 = 1 + \epsilon + \beta.$$

Now we can give the multiplication table for the representation-ring:

	1	ϵ	β	α	$\epsilon\alpha$
1	1	ϵ	β	α	$\epsilon\alpha$
ϵ	ϵ	1	β	$\epsilon\alpha$	α
β	β	β	$1+\epsilon+\beta$	$\alpha+\epsilon\alpha$	$\alpha+\epsilon\alpha$
α	α	$\epsilon\alpha$	$\alpha+\epsilon\alpha$	$1+\beta+\alpha+\epsilon\alpha$	$\epsilon+\beta+\alpha+\epsilon\alpha$
$\epsilon\alpha$	$\epsilon\alpha$	α	$\alpha+\epsilon\alpha$	$\epsilon+\beta+\alpha+\epsilon\alpha$	$1+\beta+\alpha+\epsilon\alpha$

5. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then

   $$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

   Answer:

   (a) *Let the simple representations of G be $\sigma_1, \ldots, \sigma_r$; and let $\chi_i(g)$ be the character of σ_i.*

   The simple characters $\chi_1, \ldots, \chi_r$ are linearly independent. For if say

   $$\rho_1\chi_1(g) + \cdots + \rho_s\chi_s(g) = 0$$

   it follows from the formula for the intertwining number that for any representation α

   $$\rho_1 I(\alpha, \sigma_1) + \cdots + \rho_r I(\alpha, \sigma_r) = 0.$$

   But on applying this with $\alpha = \sigma_i$ we deduce that $\rho_i = 0$ for each i.

   The characters are class functions:

   $$\chi(gxg^{-1}) = \chi(x).$$

   The space of class functions has dimension s, the number of classes in G. It follows that $r \leq s$.

   To prove that $r = s$, it is sufficient to show that the characters span the space of class functions.

   Suppose $g \in G$ has order e. Let $[g]$ denote the class of g, and let $C = \langle g \rangle$ be the cyclic group generated by g.

   The group C has e 1-dimensional representations $\theta_1, \ldots, \theta_e$ given by

   $$\theta_i : g \mapsto \omega^i,$$

where $\omega = e^{2\pi i/e}$.

Let

$$f(x) = \theta_0(x) + \omega^{-1}\theta_1(x) + \omega^{-2}\theta_2(x) + \cdots + \omega^{-e+1}\theta_{e-1}(x)$$

Then

$$f(g^j) = \begin{cases} e & \textit{if } j = 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Now let us "induce up" each of the characters θ_i *from* C *to* G. *We have*

$$\theta_i^G(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} \theta_i(y).$$

Let $F(x)$ *be the same linear combination of the induced characters that* $f(x)$ *was of the* θ_i. *Then*

$$F(x) = \frac{|G|}{|S||[x]|} \sum_{y \in [x] \cap C} f(y).$$

Since $f(y)$ *vanishes away from* g, *we deduce that* $F(x)$ *vanishes off the class* $[g]$, *and is non-zero on that class:*

$$F(x) \begin{cases} > 0 & \textit{if } x \in [g], \\ = 0 & \textit{if } x \notin [g]. \end{cases}$$

It follows that every class function on G *can be expressed as a linear combination of characters, and therefore as a linear combination of simple characters. Hence the number of simple characters is at least as great as the number of classes.*

We have shown therefore that the number of simple representations is equal to the number of classes.

(b) *Consider the regular representation* ρ *of* G. *We have*

$$\chi_\rho(g) = \begin{cases} |G| & \textit{if } g = e, \\ 0 & \textit{if } g \neq e. \end{cases}$$

Thus if α *is any representation of* G,

$$I(\rho, \alpha) = \chi_\alpha(e) = \dim \alpha.$$

Applying this to the simple representations $\alpha = \sigma_i$ *we deduce that*

$$\rho = (\dim \sigma_1)\sigma_1 + \cdots + (\dim \sigma_s)\sigma_s.$$

Taking dimensions on each side,

$$|G| = (\dim \sigma_1)^2 + \cdots + (\dim \sigma_s)^2.$$

(c) *Since* S_5 *has 7 classes (corresponding to the types* $1^5, 1^3 2, 1^2 3, 12^2, 14, 23, 5$*) it has 7 simple representations.*

We know that it has 2 representations of dimension 1, the trivial representation 1 and the parity representation ϵ*.*

Let ρ *be the natural 5-dimensional representation of* S_5 *(by permutation of coordinates). Then it is easy to show directly that*

$$\rho = 1 + \sigma$$

where σ *is a simple 4-dimensional representation.*

Now

$$\chi_\rho(1^3 2) = 3,$$

since a permutation of type $1^3 2$ *leaves 3 elements fixed. It follows that*

$$\chi_\sigma(1^3 2) = 2.$$

Hence

$$\epsilon\sigma \neq \sigma.$$

We now have 4 simple representations, of dimensions $1, 1, 4, 4$*. Let the other simple representations have dimensions* a, b, c*. Then*

$$a^2 + b^2 + c^2 = 120 - (1^2 + 1^2 + 4^2 + 4^2) = 86.$$

Since $m^2 \equiv 0$ *or* $1 \bmod 4$*, while* $86 \equiv 2 \bmod 4$*, it follows that 2 of* a, b, c *are odd and the other even.*

Let $t \in S_5$ *be a transposition. Then* $t^2 = 1$ *and so the eigenvalues of* t *in any representation* α *are* ± 1*. It follows that if* $\dim \alpha$ *is odd then* $\chi_\alpha(t) \neq 0$*, and therefore*

$$\epsilon\alpha \neq \alpha,$$

since the characters of these 2 representations differ at t*. In particular, representations of* S_5 *of odd dimension appear in pairs* $\sigma, \epsilon\sigma$*.*

Thus 2 of a, b, c *are equal odd numbers, say* $a = b$*; and*

$$2a^2 + c^2 = 86.$$

The only solution to this is $a = 5, c = 6$*. We conclude that the simple representations of* S_4 *have dimensions*

$$1, 1, 4, 4, 5, 5, 6.$$

6. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j + 1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

   Answer:

(a) *We know that*

   i. *if* $U \in \mathbf{SU}(2)$ *then* U *has eigenvalues*
   $$e^{\pm i\theta}\ (\theta \in \mathbb{R}).$$

   ii. *if* $X, Y \in \mathbf{GL}(n, k)$ *then*
   $$X \sim Y \implies X, Y \text{ have the same eigenvalues.}$$

   A fortiori, if $U \sim V \in \mathbf{SU}(2)$ *then* U, V *have the same eigenvalues.*

We shall show that the converse of the last result is also true, that is: $U \sim V$ *in* $\mathbf{SU}(2)$ *if and only if* U, V *have the same eigenvalues* $e^{\pm i\theta}$*, This is equivalent to proving that*

$$U \sim U(\theta) = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix},$$

ie we can find $V \in \mathbf{SU}(2)$ *such that*

$$V^{-1}UV = U(\theta).$$

To see this, let v *be an* $e^{i\theta}$*-eigenvalue of* U*. Normalise* v*, so that* $v^*v = 1$*; and let* w *be a unit vector orthogonal to* v*, ie* $w^*w = 1$, $v^*w = 0$*. Then the matrix*

$$V = (vw) \in \mathbf{Mat}(2, \mathbb{C})$$

is unitary; and

$$V^{-1}UV = \begin{pmatrix} e^{i\theta} & x \\ 0 & e^{-i\theta} \end{pmatrix}$$

But in a unitary matrix, the squares of the absolute values of each row and column sum to 1. It follows that

$$|e^{i\theta}|^2 + |x|^2 = 1 \implies x = 0,$$

ie

$$V^{-1}UV = U(\theta).$$

We only know that $V \in \mathbf{U}(2)$*, not that* $V \in \mathbf{SU}(2)$*. However*

$$V \in \mathbf{U}(2) \implies |\det V| = 1 \implies \det V = e^{i\phi}.$$

Thus

$$V' = e^{-i\phi/2}V \in \mathbf{SU}(2)$$

and still

$$(V')^{-1}UV = U(\theta).$$

To summarise: Since $U(-\theta) \sim U(\theta)$ *(by interchange of coordinates), we have shown that the conjugacy classes in* $\mathbf{SU}(2)$ *are*

$$C(\theta) \quad (0 \le \theta \le \pi),$$

where

$$C(\theta) = \{U \in \mathbf{SU}(2) : U \text{ has eigenvalues } e^{\pm i\theta}\}$$

(b) *Suppose $m \in \mathbb{N}$, Let $V(m)$ denote the space of homogeneous polynomials $P(z, w)$ in z, w. Thus $V(m)$ is a vector space over $\mathbb{C}$ of dimension $m+1$, with basis $z^m, z^{m-1}w, \ldots, w^m$.*

Suppose $U \in \mathbf{SU}(2)$. Then U acts on z, w by

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} z' \\ w' \end{pmatrix} = U \begin{pmatrix} z \\ w \end{pmatrix}.$$

This action in turn defines an action of $\mathbf{SU}(2)$ on $V(m)$:

$$P(z, w) \mapsto P(z', w').$$

We claim that the corresponding representation of $\mathbf{SU}(2)$ — which we denote by $D_{m/2}$ — is simple, and that these are the only simple (finite-dimensional) representations of $\mathbf{SU}(2)$ over $\mathbb{C}$.

To prove this, let

$$\mathbf{U}(1) \subset \mathbf{SU}(2)$$

be the subgroup formed by the diagonal matrices $U(\theta)$. The action of $\mathbf{SU}(2)$ on z, w restricts to the action

$$(z, w) \mapsto (e^{i\theta} z, e^{-i\theta} w)$$

of $\mathbf{U}(1)$. Thus in the action of $\mathbf{U}(1)$ on $V(m)$,

$$z^{m-r} w^r \mapsto e^{(m-2r)i\theta} z^{m-r} w^r,$$

It follows that the restriction of $D_{m/1}$ to $U(1)$ is the representation

$$D_{m/2} | \mathbf{U}(1) = E(m) + E(m-2) + \cdots + E(-m)$$

where $E(m)$ is the representation

$$e^{i\theta} \mapsto e^{mi\theta}$$

of $\mathbf{U}(1)$.

In particular, the character of $D_{m/2}$ is given by

$$\chi_{m/2}(U) = e^{mi\theta} + e^{(m-2} i\theta + \cdots + e^{-mi\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.

Now suppose $D_{m/2}$ is not *simple, say*

$$D_{m/2} = \alpha + \beta.$$

(We know that $D_{m/2}$ is semisimple, since $\mathbf{SU}(2)$ is compact.) Let a corresponding split of the representation space be

$$V(m) = W_1 \oplus W_2.$$

Since the simple parts of $D_{m/2} | \mathbf{U}(1)$ are distinct, the expression of $V(m)$ as a direct sum of $\mathbf{U}(1)$-spaces,

$$V(m) = \langle z^m \rangle \oplus \langle z^{m-1} w \rangle \oplus \cdots \oplus \langle w^m \rangle$$

is unique. It follows that W_1 must be the direct sum of some of these spaces, and W_2 the direct sum of the others. In particular $z^m \in W_1$ or $z^n \in W_2$, say $z^m \in W_1$. Let

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \in \mathbf{SU}(2).$$

Then

$$\begin{pmatrix} z \\ w \end{pmatrix} \mapsto \frac{1}{\sqrt{2}} \begin{pmatrix} z+w \\ -z+w \end{pmatrix}$$

under U. Hence

$$z^m \mapsto 2^{-m/2}(z+w)^m.$$

Since this contains non-zero components in each subspace $\langle z^{m-r}w^r \rangle$, it follows that

$$W_1 = V(m),$$

ie the representation $D_{m/2}$ of $\mathbf{SU}(2)$ in $V(m)$ is simple.

To see that every simple (finite-dimensional) representation of $\mathbf{SU}(2)$ is of this form, suppose α is such a representation. Consider its restriction to $\mathbf{U}(1)$. Suppose

$$\alpha|\mathbf{U}(1) = e_r E(r) + e_{r-1}E(r-1) + \cdots + e_{-r}E(-r) \quad (e_r, e_{r-1}, \ldots, e_{-r} \in \mathbb{N}).$$

Then α has character

$$\chi(U) = \chi(\theta) = e_r e^{ri\theta} + e_{r-1}e^{(r-1)i\theta} + \cdots + e_{-r}e^{-ri\theta}$$

if U has eigenvalues $e^{\pm i\theta}$.

Since $U(-\theta) \sim U(\theta)$ it follows that

$$\chi(-\theta) = \chi(\theta),$$

and so

$$e_{-i} = e_i,$$

ie

$$\chi(\theta) = e_r(e^{ri\theta} + e^{-ri\theta}) + e_{r-1}(e^{(r-1)i\theta} + e^{-(r-1)i\theta}) + \cdots.$$

It is easy to see that this is expressible as a sum of the $\chi_j(\theta)$ with integer (possibly negative) coefficients:

$$\chi(\theta) = a_0\chi_0(\theta) + a_{1/2}\chi_{1/2}(\theta) + \cdots + a_s\chi_s(\theta) \quad (a_0, a_{1/2}, \ldots, a_s \in \mathbb{Z}).$$

Using the intertwining number,

$$I(\alpha, \alpha) = a_0^2 + a_{1/2}^2 + \cdots + a_s^2$$

(since $I(D_j, D_k) = 0$). Since α is simple,

$$I(\alpha, \alpha) = 1.$$

It follows that one of the coefficients a_j is ± 1 and the rest are 0, ie

$$\chi(\theta) = \pm\chi_j(\theta)$$

for some half-integer j. But

$$\chi(\theta) = -\chi_j(\theta) \implies I(\alpha, D_j) = -I(D_j, D_j) = -1,$$

which is impossible. Hence

$$\chi(\theta) = \chi_j(\theta),$$

and so (since a representation is determined up to equivalence by its character)

$$\alpha = D_j.$$

(c) *As we saw above, if U has eigenvalues $e^{\pm i\theta}$ then*

$$\chi_j(U) = e^{2ji\theta} + e^{2(j-1}i\theta + \cdots + e^{-2ji\theta}.$$

(d) *Finally, we show that*

$$D_j D_k = D_{j+k} + D_{j+k-1} + \cdots + D_{|j-k|}.$$

It is sufficient to prove the corresponding result for the characters

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k}(\theta) + \chi_{j+k-1}(\theta) + \cdots + \chi_{|j-k|}(\theta).$$

We may suppose that $j \geq k$. We prove the result by induction on k.

If $k = 0$ the result is trivial, since $\chi_0(\theta) = 1$. If $k = 1/2$ then

$$\begin{aligned}
\chi_j(\theta)\chi_{1/2}(\theta) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right) \\
&= \left(e^{(2j+1)i\theta} + e^{-(2j-1)i\theta}\right) + \left(e^{(2j-1)i\theta} + e^{-(2j+1)i\theta}\right) \\
&= \chi_{j+1/2}(\theta) + \chi_{j-1/2}(\theta),
\end{aligned}$$

as required.

Suppose $k \geq 1$. Then

$$\chi_k(\theta) = \chi_{k-1}(\theta) + (e^{ki\theta} + e^{-ki\theta}).$$

Thus applying our inductive hypothesis,

$$\chi_j(\theta)\chi_k(\theta) = \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}).$$

But

$$\begin{aligned}
\chi_j(\theta)(e^{ki\theta} + e^{-ki\theta}) &= \left(e^{2ji\theta} + e^{2(j-1)i\theta} + e^{-2ji\theta}\right)\left(e^{ki\theta} + e^{-ki\theta}\right) \\
&= \chi_{j+k}(\theta) + \chi j - k(\theta),
\end{aligned}$$

giving the required result

$$\begin{aligned}
\chi_j(\theta)\chi_k(\theta) &= \chi_{j+k-1}(\theta) + \cdots + \chi_{j-k+1} + \chi_{j+k}(\theta) + \chi j - k(\theta) \\
&= \chi_{j+k}(\theta) + \cdots + \chi_{j-k}.
\end{aligned}$$

7. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & \text{-1} \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & \text{-1} \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & \text{-1} \\ 1 & 1 \end{pmatrix}.$$

   Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

   Which of the above 5 matrices X are themselves expressible in the form $X = e^Y$ for some real matrix Y? (Justify your answers in all cases.)

   Answer:

   (a) *The exponential of a square matrix X is defined by*

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots.$$

   [It is a straightforward matter to verify that this series always converges, using the fact that the matrix norm defined by

$$\|X\|^2 = \mathbf{tr}(X'X) = \sum_{i,j} |x_{ij}|$$

   has the properties

   i. $\|X + Y\| \le \|X\| + \|Y\|$,

   ii. $\|\lambda X\| = |\lambda|\|X\|$,

   iii. $\|XY\| \le \|X\|\|Y\|$.

   However, I would not require this observation.]

   i. If

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

   then

$$e^X = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}$$

   ii. If

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   then

$$X^2 = I,$$

   and so

$$\begin{aligned} e^X &= (1 + 1/2! + 1/4! + \cdots)I + (1/1! + 1/3! + \cdots)X \\ &= \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix} \end{aligned}$$

iii. *If*

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then

$$X^2 = -I,$$

and so

$$\begin{aligned} e^X &= (1 - 1/2! + 1/4! - \cdots)I + (1/1! - 1/3! + \cdots)X \\ &= \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \end{aligned}$$

iv. *If*

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + J,$$

where

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

then (since I, J commute)

$$\begin{aligned} e^X &= e^I e^J \\ &= \begin{pmatrix} e\cos 1 & -e\sin 1 \\ e\sin 1 & e\cos 1 \end{pmatrix} \end{aligned}$$

(b) *i.* *If*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

then

$$\det X = -1,$$

and so $X \neq e^Y$ since

$$\det e^Y = e^{\mathbf{tr} Y} > 0.$$

ii. *If*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

then again

$$\det X < 0,$$

and so $X \neq e^Y$.

iii. *By the isomorphims between the complex numbers $x + iy$ and the matrices*

$$\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

we see that

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

corresponds to i.

Since

$$i = e^{\pi i/2}$$

we see that

$$J = e^Y$$

with

$$\begin{aligned} Y &= \frac{\pi}{2} J \\ &= \begin{pmatrix} 0 & -\pi/2 \\ \pi & 0 \end{pmatrix}. \end{aligned}$$

iv. *Similarly*

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

corresponds to the complex number $1 + i$. *Since*

$$\begin{aligned} 1 + i &= \sqrt{2} e^{\pi i/4} \\ &= e^{\log 2/2 + \pi i/4} \end{aligned}$$

$X = e^Y$ *with*

$$Y = \begin{pmatrix} \log 2/2 & -\pi/4 \\ \pi/4 & \log 2/2 \end{pmatrix}.$$

8. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathcal{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

   $$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}), \mathbf{GL}(n, \mathbb{C}), \mathbf{SL}(n, \mathbb{C})?$$

   Answer:

   (a) *A linear group is a closed subgroup* $G \subset \mathbf{GL}(n, \mathbb{R})$ *for some* n.

   (b) *A Lie algebra is defined by giving*

      i. *a vector space* L;

      ii. *a binary operation on* L, *ie a map*

      $$L \times L \to L : \ (X, Y) \mapsto [X, Y]$$

   satisfying the conditions

      i. *The product* $[X, Y]$ *is bilinear in* X, Y;

      ii. *The product is skew-symmetric:*

      $$[Y, X] = -[X, Y];$$

      iii. *Jacobi's identity is satisfied:*

      $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$$

      for all $X, Y, Z \in L$.

(c) *Suppose $G \subset \mathbf{GL}(n, \mathbb{R})$ is a linear group. Then its Lie algebra $L = \mathcal{L}G$ is defined to be*

$$L = \{X \in \mathbf{Mat}(n, \mathbb{R}) : e^{tX} \in G \; \forall t \in \mathbb{R}\}.$$

(d) *It follows at once from this definition that*

$$X \in L, \; \lambda \in R \implies \lambda X \in L.$$

Thus to see that L is a vector subspace of $\mathbf{Mat}(n, \mathbb{R})$ we must show that

$$X, Y \in L \implies X + Y \in L.$$

Now

$$\left(e^{X/n} e^{Y/n}\right)^n \mapsto e^{X+Y}$$

as $n \mapsto \infty$. (This can be seen by taking the logarithms of each side.) It follows that

$$X, Y \in L \implies e^{X+Y} \in G.$$

On replacing X, Y by tX, tY we see that

$$\begin{aligned} X, Y \in L &\implies e^{t(X+Y)} \in G \\ &\implies X + Y \in L. \end{aligned}$$

Similarly

$$\left(e^{X/n} e^{Y/n} e^{-X/n} e^{-Y/n}\right)^{n^2} \mapsto e^{[X,Y]},$$

as may be seen again on taking logarithms. It follows that

$$X, Y \in L \implies e^{[X,Y]} \in G.$$

Taking tX in place of X, this implies that

$$\begin{aligned} X, Y \in L &\implies e^{t[X,Y]} \in G \\ &\implies [X, Y] \in L. \end{aligned}$$

Thus L is a Lie algebra.

(e) *The dimension of a linear group G is the dimension of the real vector space $\mathcal{L}G$:*

$$\dim G = \dim_{\mathbb{R}} \mathcal{L}G.$$

(f) i. *We have*

$$\mathbf{o}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0\}$$

A skew symmetric matrix X is determined by giving the entries above the diagonal. This determines the entries below the diagonal; while those on the diagonal are 0. Thus

$$\dim O(n) = \dim o(n) = \frac{n(n-1)}{2}.$$

ii. *We have*

$$\mathbf{so}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0, \mathbf{tr} X = 0\} = \o(n).$$

Since $X' + X = 0 \implies \mathbf{tr} X = 0$,

$$\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}.$$

iii. *We have*

$$\mathbf{u}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0\}$$

Again, the elements above the diagonal determine those below the diagonal; while those on the diagonal are purely imaginary. Thus

$$\begin{aligned}\dim \mathbf{U}(n) &= 2\frac{n(n-1)}{2} + n \\ &= n^2.\end{aligned}$$

iv. *We have*

$$\mathbf{su}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0,\ \mathbf{tr} X = 0\}$$

This gives one linear condition on the (purely imaginary) diagonal elements. Thus

$$\dim \mathbf{SU}(n) = \dim \mathbf{U}(n) - 1 = n^2 - 1.$$

v. *We have*

$$\mathbf{gl}(n, \mathbb{R}) = \mathbf{Mat}(n, \mathbb{R}).$$

Thus

$$\dim \mathbf{GL}(n, \mathbb{R}) = n^2.$$

vi. *We have*

$$\mathbf{sl}(n, \mathbb{R}) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : \mathbf{tr} X = 0\}.$$

This imposes one linear condition on X*. Thus*

$$\dim \mathbf{SL}(n, \mathbb{R}) = \dim \mathbf{GL}(n, \mathbb{R}) - 1 = n^2 - 1.$$

vii. *We have*

$$\mathbf{gl}(n, \mathbb{C}) = \mathbf{Mat}(n, \mathbb{C}).$$

Each of the n^2 *complex entries takes 2 real values. Thus*

$$\dim \mathbf{GL}(n, \mathbb{C}) = 2n^2.$$

viii. *We have*

$$\mathbf{sl}(n, \mathbb{C}) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : \mathbf{tr} X = 0\}.$$

This imposes one complex linear condition on X*, or 2 real linear conditions. Thus*

$$\dim \mathbf{SL}(n, \mathbb{C}) = \dim \mathbf{GL}(n, \mathbb{C}) - 1 = 2n^2 - 2.$$

9. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

   Answer:

   (a) We have

$$\begin{aligned}
\mathbf{u}(2) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) : \; e^{tX} \in \mathbf{U}(2)\ \forall t \in \mathbb{R}\} \\
&= \{X : (e^{tX})^* e^{tX} = I\ \forall t\} \\
&= \{X : e^{tX^*} = e^{-tX}\ \forall t\} \\
&= \{X : X^* = -X\} \\
&= \{X : X^* + X = 0\},
\end{aligned}$$

while

$$\begin{aligned}
\mathbf{sl}(2,\mathbb{C}) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) : \; e^{tX} \in \mathbf{SL}(2,\mathbb{C})\ \forall t \in \mathbb{R}\} \\
&= \{X : \det e^{tX} = 1\ \forall t\} \\
&= \{X : e^{t\mathbf{tr}X} = 1\ \forall t\} \\
&= \{X : \mathbf{tr}X = 0\}.
\end{aligned}$$

Since

$$\mathbf{SU}(2) = \mathbf{U}(2) \cap \mathbf{SL}(2,\mathbb{C})$$

it follows that

$$\begin{aligned}
\mathbf{su}(2) &= \mathbf{u}(2) \cap \mathbf{sl}(2,\mathbb{C}) \\
&= \{X : X^* + X = 0,\ \mathbf{tr}X = 0\}.
\end{aligned}$$

The 3 matrices

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{su}(2)$.
We have

$$\begin{aligned}
[e,f] &= ef - fe = -2g, \\
[e,g] &= eg - ge = 2f, \\
[f,g] &= fg - gf = -2e
\end{aligned}$$

Thus

$$\mathbf{su}(2) = \langle e, f, g : [e,f] = -2g, [e,g] = 2f, [f,g] = -2e \rangle.$$

(b) We have

$$\begin{aligned}
\mathbf{o}(3) &= \{X \in \mathbf{Mat}(3,\mathbb{R}) : \; e^{tX} \in \mathbf{O}(3)\ \forall t \in \mathbb{R}\} \\
&= \{X : (e^{tX})' e^{tX} = I\ \forall t\} \\
&= \{X : e^{tX'} = e^{-tX}\ \forall t\} \\
&= \{X : X' = -X\} \\
&= \{X : X' + X = 0\},
\end{aligned}$$

while

$$\begin{aligned}\mathbf{sl}(3,\mathbb{R}) &= \{X \in \mathbf{Mat}(3,\mathbb{R}) : \ e^{tX} \in \mathbf{SL}(3,\mathbb{R})\ \forall t \in \mathbb{R}\}\\ &= \{X : \det e^{tX} = 1\ \forall t\}\\ &= \{X : e^{t\mathbf{tr}X} = 1\ \forall t\}\\ &= \{X : \mathbf{tr}X = 0\}.\end{aligned}$$

Since

$$\mathbf{SO}(3) = \mathbf{O}(3) \cap \mathbf{SL}(3,\mathbb{R})$$

it follows that

$$\begin{aligned}\mathbf{so}(3) &= \mathbf{o}(3) \cap \mathbf{sl}(3,\mathbb{R})\\ &= \{X : X' + X = 0,\ \mathbf{tr}X = 0\}\\ &= \{X : X' + X = 0\}\end{aligned}$$

since a skew-symmetric matrix necessarily has trace 0.
The 3 matrices

$$U = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix},\ V = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ -1 & 0 & 0\end{pmatrix},\ W = \begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

form a basis for the vector space $\mathbf{so}(3)$*.*
We have

$$[V,W] = U;$$

and so by cyclic permutation of indices (or coordinates)

$$[W,U] = V,\ [U,V] = W.$$

Thus

$$\mathbf{so}(3) = \langle U,V,W : [U,V] = W, [U,W] = -V, [V,W] = U\rangle.$$

(c) *The Lie algebras* $su(2)$ *and* $so(3)$ *are isomorphic under the correspondence*

$$e \leftrightarrow -2U,\ f \leftrightarrow -2V,\ g \leftrightarrow -2W.$$

(d) *However, the groups* $\mathbf{SU}(2), \mathbf{SO}(3)$ *are not isomorphic, since*

$$Z\mathbf{SU}(2) = \{\pm I\} \text{ while } Z\mathbf{SO}(3) = \{I\}.$$

10. Define a *representation* of a Lie algebra $\mathcal{L}$. What is meant by saying that such a representation is (a) *simple*, (b) *semisimple*?

Determine the Lie algebra of $\mathbf{SL}(2,\mathbb{R})$, and find all the simple representations of this algebra.

Show that every representation of the group $\mathbf{SL}(2,\mathbb{R})$ is semisimple, stating carefully but without proof any results you need.

Answer:

(a) *Suppose L is a real Lie algebra. A representation of L in the complex vector space V is defined by giving a map*

$$L \times V \to V : \ (X, v) \mapsto Xv$$

which is bilinear over $\mathbb{R}$ and which satisfies the condition

$$[X, Y]v = X(Yv) - Y(Xv)$$

for all $X, Y \in L, v \in V$.

A representation of L in V is thus the same as a representation of the complexification $L_{\mathbb{C}}$ of L in V.

(b) *The representation is said to be simple if no proper subspace $U \subset V$ is stable under L.*

It is said to be semisimple if V is a direct sum of simple subspaces.

(c) *We have*

$$\begin{aligned}\mathbf{sl}(2, \mathbb{R}) &= \{X \in \mathbf{Mat}(2, \mathbb{R}) : \ e^{tX} \in \mathbf{SL}(2, \mathbb{R}) \forall t \in \mathbb{R}\} \\ &= \{X : \det e^{tX} = 1 \forall t\} \\ &= \{X : e^{t\mathbf{tr}X} = 1 \forall t\} \\ &= \{X : \mathbf{tr}X = 0\}.\end{aligned}$$

The 3 matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

form a basis for the vector space $\mathbf{sl}(2, \mathbb{R})$.

We have

$$[H, E] = 2E, \ [H, F] = -2F, \ [E, F] = H.$$

Thus

$$\mathbf{sl}(2, \mathbb{R}) = \langle H, E, F : [H, E] = 2E, [H, F] = -2F, [V, W] = H \rangle.$$

(d) *Suppose we have a simple representation of $\mathbf{sl}(2, \mathbb{R})$ on V. Let v be an eigenvector of H with eigenvalue λ:*

$$Hv = \lambda v.$$

Then

$$[H, E]v = 2Ev,$$

ie

$$HEv - EHv = 2Ev.$$

In other words, since $Hv = \lambda v$,

$$H(Ev) = (\lambda + 2)Ev,$$

that is, Ev is an eigenvector of H with eigenvalue $\lambda + 2$.

By the same argument $E^2v, E^3v, \ldots$ are all eigenvectors of H with eigenvalues $\lambda+4, \lambda+6, \ldots,$ at least until they vanish.

This must happen at some point, since V is finite-dimensional; say

$$E^{r+1}v = 0, E^r v \neq 0.$$

Similarly we find that

$$Fv, F^2v, \ldots$$

are also eigenvectors of H (until they vanish) with eigenvalues $\lambda-2, \lambda-4, \ldots$. Again we must have

$$F^{s+1}v = 0, F^s v \neq 0$$

for some s.

Now let us write e_0 for $F^s v$, so that

$$Fe_0 = 0;$$

and let $\mu = \lambda - 2s$, so that e_0 is a μ-eigenvector of H:

$$He_0 = \mu e_0.$$

Set

$$e_i = E^i e_0,$$

so that

$$e_{i+1} = Ee_i$$

for $i \geq 0$. Thus e_i is a $(\mu+2i)$-eigenvector of H; and if $m = r+s$ then

$$e_m \neq 0,\ e_{m+1} = Ee_m = 0.$$

By the argument above, Fe_i is an eigenvector of H with the same eigenvalue, $\mu + 2(i-1)$, as e_{i-1}. We shall show by induction on i that Fe_i is a scalar multiple of e_{i-1}, say

$$Fe_i = \rho_i e_{i-1}.$$

The result holds for $i = 0$ with $\rho_0 = 0$. Suppose it holds for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned} Fe_{m+1} &= FEe_m \\ &= (EF - [E,F])e_m \\ &= \rho_m Ee_{m-1} - He_m \\ &= (\rho_m - \mu - 2m)e_m. \end{aligned}$$

This proves the result, and also shows that

$$\rho_{i+1} = \rho_i - \mu - 2i$$

for each i. It follows that

$$\rho_i = -i\mu - i(i-1).$$

We must have $\rho_{m+1} = 0$. Hence

$$\mu = m.$$

We conclude that the subspace

$$\langle e_0, \ldots, e_m \rangle$$

is stable under $\mathbf{sl}(2, \mathbb{R})$, and so must be the whole of V.

Thus we have shown that there is at most 1 simple representation of each dimension n, and we have determined this explicitly, if it exists. In fact it is a straightforward matter to verify that the above actions of H, E, F on $\langle e_0, \ldots, e_m \rangle$ do indeed define a representation of $\mathbf{sl}(2, \mathbb{R})$; so that this Lie algebra has exactly 1 simple representation of each dimension.

(e) i. *The representations over $\mathbb{C}$ of a real Lie algebra L correspond to the representations of its complexification $\mathbb{C}L$.*

ii. *Thus if two real Lie algebras have isomorphic complexifications then there is a one-one correspondence between their representations, with simple or semisimple representations of one corresponding to simple or semisimple representations of the other.*

iii. *The Lie algebras $\mathbf{sl}(n, \mathbb{R})$ and $\mathbf{su}(n)$ have the same complexification:*

$$\mathbb{C}\mathbf{sl}(n, \mathbb{R}) = \mathbf{sl}(c, \mathbb{C}) = \mathbb{C}\mathbf{su}(n).$$

iv. *The representations of $\mathbf{su}(2)$ can all be lifted to representations of $\mathbf{SU}(2)$, since $\mathbf{SU}(2) \cong S^3$ is simply-connected.*

v. *The representations of $\mathbf{SU}(2)$ are all semisimple, since $\mathbf{SU}(2)$ is compact.*

vi. *Hence the representations of $\mathbf{su}(2$ are all semisimple, and so the representations of $\mathbf{sl}(2, \mathbb{R})$ are all semisimple.*