# Course 424

# Group Representations III

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*Answer as many questions as you can; all carry the same number of marks.*
*In this exam, 'Lie algebra' means* Lie algebra over $\mathbb{R}$*, and 'representation' means* finite-dimensional representation over $\mathbb{C}$*.*

1. Define the *exponential* $e^X$ of a square matrix $X$.

   Determine $e^X$ in each of the following cases:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

   Which of these 6 matrices $X$ are themselves expressible in the form $X = e^Y$, where $Y$ is a real matrix? (Justify your answers in all cases.)

   **Answer:** *The exponential of a square matrix $X$ is defined by*

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots.$$

   *This series converges for all $X \in \mathbf{Mat}(n,k)$ by comparison with the series for $e^{\|X\|}$, since $\|X^n\| \leq \|X\|^n$.*

(a) *If*

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*then*

$$X^2 = 0$$

*and so*

$$e^X = I + X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

(b) *If*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

*then*

$$X^2 = I,$$

*and so*

$$e^X = \begin{pmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{pmatrix}$$

(c) *If*

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

*then*

$$X^2 = -I,$$

*and so*

$$e^X = \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix}.$$

(d) *If*

$$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

*then*

$$X = \begin{pmatrix} e & 0 \\ 0 & e^{-1} \end{pmatrix}.$$

(e) *If*

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = I + Y,$$

*where*

$$Y = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

*then, since $I, Y$ commute,*

$$e^X = e^I e^Y = \begin{pmatrix} e\cos 1 & -e\sin 1 \\ e\sin 1 & e\cos 1 \end{pmatrix}.$$

*(f) If*

$$X = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = -I + Z,$$

*where*

$$Z = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

*then, since $-I, Z$ commute,*

$$e^X = e^{-I} e^Z = \begin{pmatrix} e^{-1} & -e^{-1} \\ 0 & e^{-1} \end{pmatrix}.$$

*(a) $e^Y$ is non-singular for all $Y$, since $e^Y e^{-Y} = I$. Since $X$ is singular in this case, $X \neq e^Y$.*

*(b) $X$ has eigenvalues $\pm 1$. Suppose $X = e^Y$; and suppose $Y$ has eigenvalues $\lambda, \mu$. Then $X$ has eigenvalues $e^\lambda, e^\mu$. There are two possibilities. Either $\lambda, \mu$ are complex conjugates, in which case the same is true of $e^\lambda, e^\mu$; or else $\lambda, \mu$ are both real, in which case $e^\lambda, e^\mu > 0$. In neither case can we get $\pm 1$. Hence $X \neq e^Y$.*

*(c) By the isomorphism between the complex numbers $z = x + iy$ and the matrices*

$$\mathbb{C}(z) = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}$$

*we see that*

$$X = \mathbb{C}(i).$$

*Since*

$$i = e^{\pi i/2},$$

*while*

$$\mathbb{C}(e^z) = e^{\mathbb{C}(z)},$$

*it follows that $X = e^Y$ with*

$$Y = \begin{pmatrix} 0 & -\pi/2 \\ \pi/2 & 0 \end{pmatrix}.$$

*(d) $X$ has eigenvalues $\pm 1$. Thus by the argument in case (b) above, $X \neq e^Y$.*

(e) *We have*

$$X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \mathbb{C}(1+i).$$

*But*

$$1 + i = \sqrt{2}e^{\pi i/4} = e^{\log 2/2 + \pi i/4}.$$

*Thus* $X = e^Y$ *with*

$$Y = X = \begin{pmatrix} \log 2/2 & -\pi/4 \\ \pi/4 & \log 2/2 \end{pmatrix}.$$

(f) *$X$ has eigenvalues $-1, -1$. Thus if $X = e^Y$ (with $Y$ real) then $Y$ must have eigenvalues $\pm(2n+1)\pi i$ for some integer $n$. In particular, $Y$ has distinct eigenvalues, and so is semisimple (diagonalisable over $\mathbb{C}$).*

*But in that case $X = e^Y$ would also be semisimple. That is impossible, since a diagonalisable matrix with eigenvalues $-1, -1$ is necessarily $-I$. Hence $X \neq e^Y$.*

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group $G$, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n,\mathbb{R}), \mathbf{SL}(n,\mathbb{R}), \mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C})?$$

**Answer:** *A linear group is a closed subgroup $G \subset \mathbf{GL}(n,\mathbb{R})$ for some $n$.*

*A Lie algebra is defined by giving*

(a) *a vector space $L$;*

(b) *a binary operation on $L$, ie a map*

$$L \times L \to L : \ (X,Y) \mapsto [X,Y]$$

*satisfying the conditions*

(a) *The product $[X,Y]$ is bilinear in $X, Y$;*

(b) *The product is skew-symmetric:*

$$[Y,X] = -[X,Y];$$

(c) *Jacobi's identity is satisfied:*

$$[[X,Y],Z]+[[Y,Z],X]+[[Z,X],Y]=0$$

*for all* $X,Y,Z \in L$.

*Suppose* $G \subset \mathbf{GL}(n,\mathbb{R})$ *is a linear group. Then its Lie algebra* $L=\mathcal{L}G$ *is defined to be*

$$L=\{X \in \mathbf{Mat}(n,\mathbb{R}) : e^{tX} \in G \forall t \in \mathbb{R}\}.$$

*It follows at once from this definition that*

$$X \in L,\ \lambda \in R \implies \lambda X \in L.$$

*Thus to see that* $L$ *is a vector subspace of* $\mathbf{Mat}(n,\mathbb{R})$ *we must show that*

$$X,Y \in L \implies X+Y \in L.$$

*Now*

$$\left(e^{X/n}e^{Y/n}\right)^n \mapsto e^{X+Y}$$

*as* $n \mapsto \infty$. *(This can be seen by taking the logarithms of each side.) It follows that*

$$X,Y \in L \implies e^{X+Y} \in G.$$

*On replacing* $X,Y$ *by* $tX,tY$ *we see that*

$$\begin{aligned} X,Y \in L &\implies e^{t(X+Y)} \in G \\ &\implies X+Y \in L. \end{aligned}$$

*Similarly*

$$\left(e^{X/n}e^{Y/n}e^{-X/n}e^{-Y/n}\right)^{n^2} \mapsto e^{[X,Y]},$$

*as may be seen again on taking logarithms. It follows that*

$$X,Y \in L \implies e^{[X,Y]} \in G.$$

*Taking* $tX$ *in place of* $X$, *this implies that*

$$\begin{aligned} X,Y \in L &\implies e^{t[X,Y]} \in G \\ &\implies [X,Y] \in L. \end{aligned}$$

*Thus* $L$ *is a Lie algebra.*

*The dimension of a linear group* $G$ *is the dimension of the real vector space* $\mathscr{L}G$:

$$\dim G = \dim_{\mathbb{R}} \mathscr{L}G.$$

(a) *We have*

$$\mathbf{o}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0\}$$

*A skew symmetric matrix $X$ is determined by giving the entries above the diagonal. This determines the entries below the diagonal; while those on the diagonal are 0. Thus*

$$\dim O(n) = \dim o(n) = \frac{n(n-1)}{2}.$$

(b) *We have*

$$\mathbf{so}(n) = \{X \in \mathbf{Mat}(n, \mathbb{R}) : X' + X = 0, \operatorname{tr} X = 0\} = \mathbf{o}(n),$$

*since* $X' + X = 0 \implies \operatorname{tr} X = 0$. *Thus*

$$\dim SO(n) = \dim O(n) = \frac{n(n-1)}{2}.$$

(c) *We have*

$$\mathbf{u}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0\}$$

*Again, the elements above the diagonal determine those below the diagonal; while those on the diagonal are purely imaginary. Thus*

$$\begin{aligned}\dim \mathbf{U}(n) &= 2\frac{n(n-1)}{2} + n \\ &= n^2.\end{aligned}$$

(d) *We have*

$$\mathbf{su}(n) = \{X \in \mathbf{Mat}(n, \mathbb{C}) : X^* + X = 0,\ \operatorname{tr} X = 0\}$$

*This gives one linear condition on the (purely imaginary) diagonal elements. Thus*

$$\dim \mathbf{SU}(n) = \dim \mathbf{U}(n) - 1 = n^2 - 1.$$

(e) *We have*

$$\mathbf{gl}(n, \mathbb{R}) = \mathbf{Mat}(n, \mathbb{R}).$$

*Thus*

$$\dim \mathbf{GL}(n, \mathbb{R}) = n^2.$$

(f) *We have*

$$\mathbf{sl}(n,\mathbb{R}) = \{X \in \mathbf{Mat}(n,\mathbb{R}) : \operatorname{tr} X = 0\}.$$

*This imposes one linear condition on $X$. Thus*

$$\dim \mathbf{SL}(n,\mathbb{R}) = \dim \mathbf{GL}(n,\mathbb{R}) - 1 = n^2 - 1.$$

(g) *We have*

$$\mathbf{gl}(n,\mathbb{C}) = \mathbf{Mat}(n,\mathbb{C}).$$

*Each of the $n^2$ complex entries takes 2 real values. Thus*

$$\dim \mathbf{GL}(n,\mathbb{C}) = 2n^2.$$

(h) *We have*

$$\mathbf{sl}(n,\mathbb{C}) = \{X \in \mathbf{Mat}(n,\mathbb{C}) : \operatorname{tr} X = 0\}.$$

*This imposes one complex linear condition on $X$, or 2 real linear conditions. Thus*

$$\dim \mathbf{SL}(n,\mathbb{C}) = \dim \mathbf{GL}(n,\mathbb{C}) - 1 = 2n^2 - 2.$$

3. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

   **Answer:** *We have*

$$\begin{aligned}
\mathbf{u}(2) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) :\ e^{tX} \in \mathbf{U}(2) \forall t \in \mathbb{R}\} \\
&= \{X : (e^{tX})^* e^{tX} = I \forall t\} \\
&= \{X : e^{tX^*} = e^{-tX} = I \forall t\} \\
&= \{X : X^* = -X\} \\
&= \{X : X^* + X = 0\}.
\end{aligned}$$

*Also*

$$\begin{aligned}
\mathbf{sl}(2,\mathbb{C}) &= \{X \in \mathbf{Mat}(2,\mathbb{C}) :\ e^{tX} \in \mathbf{SL}(2,\mathbb{C}) \forall t \in \mathbb{R}\} \\
&= \{X : \det e^{tX} = 1 \forall t\} \\
&= \{X : e^{t \operatorname{tr} X} = 1 \forall t\} \\
&= \{X : \operatorname{tr} X = 0\}.
\end{aligned}$$

*Since*

$$\mathbf{SU}(2) = \mathbf{U}(2) \cap \mathbf{SL}(2, \mathbb{C})$$

*it follows that*

$$\begin{aligned}\mathbf{su}(2) &= \mathbf{u}(2) \cap \mathbf{sl}(2, \mathbb{C}) \\ &= \{X : X^* + X = 0,\ \operatorname{tr} X = 0\}.\end{aligned}$$

*The 3 matrices*

$$e = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},\ f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ g = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

*form a basis for the vector space* $\mathbf{su}(2)$*.*

*We have*

$$\begin{aligned}[e, f] &= ef - fe = -2g, \\ [e, g] &= eg - ge = 2f, \\ [f, g] &= fg - gf = -2e\end{aligned}$$

*Thus*

$$\mathbf{su}(2) = \langle e, f, g : [e, f] = -2g, [e, g] = 2f, [f, g] = -2e \rangle.$$

*We have*

$$\begin{aligned}\mathbf{o}(3) &= \{X \in \mathbf{Mat}(3, \mathbb{R}) :\ e^{tX} \in \mathbf{O}(3) \forall t \in \mathbb{R}\} \\ &= \{X : (e^{tX})' e^{tX} = I \forall t\} \\ &= \{X : e^{tX'} = e^{-tX} = I \forall t\} \\ &= \{X : X' = -X\} \\ &= \{X : X' + X = 0\}.\end{aligned}$$

*Also*

$$\begin{aligned}\mathbf{sl}(3, \mathbb{R}) &= \{X \in \mathbf{Mat}(3, \mathbb{R}) :\ e^{tX} \in \mathbf{SL}(3, \mathbb{R}) \forall t \in \mathbb{R}\} \\ &= \{X : \det e^{tX} = 1 \forall t\} \\ &= \{X : e^{t \operatorname{tr} X} = 1 \forall t\} \\ &= \{X : \operatorname{tr} X = 0\}.\end{aligned}$$

*Since*

$$\mathbf{SO}(3) = \mathbf{O}(3) \cap \mathbf{SL}(3, \mathbb{R})$$

*it follows that*

$$\begin{aligned}\mathbf{so}(3) &= \mathbf{o}(3) \cap \mathbf{sl}(3, \mathbb{R}) \\ &= \{X : X' + X = 0,\ \operatorname{tr} X = 0\} \\ &= \{X : X' + X = 0\}\end{aligned}$$

*since a skew-symmetric matrix necessarily has trace 0.*

*The 3 matrices*

$$U = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ V = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},\ W = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

*form a basis for the vector space* $\mathbf{so}(3)$*.*

*We have*

$$[V, W] = U;$$

*and so by cyclic permutation of indices (or coordinates)*

$$[W, U] = V,\ [U, V] = W.$$

*Thus*

$$\mathbf{so}(3) = \langle U, V, W : [U, V] = W, [U, W] = -V, [V, W] = U \rangle.$$

*Finally,* $su(2)$ *and* $so(3)$ *are isomorphic under the correspondence*

$$e \leftrightarrow -2U,\ f \leftrightarrow -2V,\ g \leftrightarrow -2W.$$

*However, the groups* $\mathbf{SU}(2), \mathbf{SO}(3)$ *are not isomorphic, since*

$$Z\mathbf{SU}(2) = \{\pm I\} \text{ while } Z\mathbf{SO}(3) = \{I\}.$$

4. Define a *representation* of a Lie algebra; and show how each representation $\alpha$ of a linear group $G$ gives rise to a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$.

   Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$; and show that this Lie algebra $\mathbf{sl}(2, \mathbb{R})$ has just 1 simple representation of each dimension $1, 2, 3, \ldots$.

   **Answer:** *Suppose* $L$ *is a real Lie algebra. A representation of* $L$ *in the complex vector space* $V$ *is defined by giving a map*

$$L \times V \to V : (X, v) \mapsto Xv$$

*which is bilinear over* $\mathbb{R}$ *and which satisfies the condition*

$$[X, Y]v = X(Yv) - Y(Xv)$$

*for all* $X, Y \in L, v \in V$.

*A representation of* $L$ *in* $V$ *is thus the same as a representation of the complexification* $L_{\mathbb{C}}$ *of* $L$ *in* $V$.

*Suppose* $\alpha$ *is a representation of the linear group* $G$*, ie a homomorphism*

$$\alpha : G \to \mathbf{GL}(n, \mathbb{C}).$$

*Under the Lie correspondence this gives rise to a Lie algebra homomorphism*

$$A = \mathscr{L}\alpha : L = \mathscr{L}G \to \mathbf{gl}(n, \mathbb{C}).$$

*But now* $L$ *acts on* $V = C^n$ *by*

$$Xv = A(X)v.$$

*This defines a representation of* $L$ *in* $V$ *since*

$$\begin{aligned}
[X, Y]v &= A([X, Y])v \\
&= [AX, AY]v \\
&= ((AX)(AY) - (AY)(AX))v \\
&= X(Yv) - Y(Xv).
\end{aligned}$$

*We have*

$$\begin{aligned}
\mathbf{sl}(2, \mathbb{R}) &= \{X \in \mathbf{Mat}(2, \mathbb{R}) :\ e^{tX} \in \mathbf{SL}(2, \mathbb{R}) \forall t \in \mathbb{R}\} \\
&= \{X : \det e^{tX} = 1 \forall t\} \\
&= \{X : e^{t \operatorname{tr} X} = 1 \forall t\} \\
&= \{X : \operatorname{tr} X = 0\}.
\end{aligned}$$

*The 3 matrices*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

*form a basis for the vector space* $\mathbf{sl}(2, \mathbb{R})$.

*We have*

$$[H, E] = 2E,\ [H, F] = -2F,\ [E, F] = H.$$

*Thus*

$$\mathbf{sl}(2,\mathbb{R}) = \langle H, E, F : [H,E] = 2E, [H,F] = -2F, [V,W] = H\rangle.$$

*Now suppose we have a simple representation of* $\mathbf{sl}(2,\mathbb{R})$ *on* $V$*. Suppose* $v$ *is an eigenvector of* $H$ *with eigenvalue* $\lambda$*:*

$$Hv = \lambda v.$$

*Now*

$$[H,E]v = 2Ev,$$

*that is,*

$$HEv - EHv = 2Ev.$$

*In other words, since* $Hv = \lambda v$*,*

$$H(Ev) = (\lambda + 2)Ev,$$

*ie* $Ev$ *is an eigenvector of* $H$ *with eigenvalue* $\lambda + 2$*.*

*By the same argument* $E^2v, E^3v, \ldots$ *are all eigenvectors of* $H$ *with eigenvalues* $\lambda + 4, \lambda + 6, \ldots,$ *at least until they vanish.*

*This must happen at some point, since* $V$ *is finite-dimensional; say*

$$E^{r+1}v = 0, E^r v \neq 0.$$

*Similarly we find that*

$$Fv, F^2v, \ldots$$

*are also eigenvectors of* $H$ *(until they vanish) with eigenvalues* $\lambda - 2, \lambda - 4, \ldots$*. Again we must have*

$$F^{s+1}v = 0, F^s v \neq 0$$

*for some* $s$*.*

*Now let us write* $e_0$ *for* $F^s v$*, so that*

$$Fe_0 = 0;$$

*and let us set*

$$e^i = E^i e_0.$$

*Then the $e_i$ are all eigenvectors of $H$. Let us set $e_i = 0$ for $i$ outside the range $[0, n-1]$. Suppose $e_0$ is a $\mu$-eigenvector. Then $e_i$ is a $(\mu + 2i)$-egenvector. Let us suppose that there are $n$ eigenvectors in the sequence, ie*

$$e_{n-1} \neq 0, \ Ee_{n-1} = 0.$$

*Now we show by induction that*

$$Fe_i = \rho_i e_{i-1}$$

*for each $i$. The result holds for $i = 0$ with $\rho_0 = 0$. Suppose it holds for $i = 1, 2, \ldots, m$. Then*

$$\begin{aligned} Fe_{m+1} &= FEe_m \\ &= (EF - [E, F])e_m \\ &= \rho_m Ee_{m-1} - He_m \\ &= (\rho_m - \mu - 2m)e_m. \end{aligned}$$

*This proves the result, and also shows that*

$$\rho_{i+1} = \rho_i - \mu - 2i$$

*for each $i$. It follows that*

$$\rho_i = -i\mu - i(i-1).$$

*We must have $\rho_n = 0$. Hence*

$$\mu = n - 1.$$

*We conclude that the subspace*

$$\langle e_0, \ldots, e_{n-1} \rangle$$

*is stable under $\mathbf{sl}(2, \mathbb{R})$, and so must be the whole of $V$.*

*Thus we have shown that there is at most 1 simple representation of each dimension $n$, and we have determined this explicitly, if it exists. In fact it is a straightforward matter to verify that the above actions of $H, E, F$ on $\langle e_0, \ldots, e_{n-1} \rangle$ do indeed define a representation of $\mathbf{sl}(2, \mathbb{R})$; so that this Lie algebra has exactly 1 simple representation of each dimension.*

5. Show that a compact connected abelian linear group of dimension $n$ is necessarily isomorphic to the torus $\mathbb{T}^n$.

**Answer:** *If $G$ is a abelian linear group then $\mathscr{L}G$ is trivial, ie $[X, Y] = 0$ for all $X, Y \in \mathscr{L}G$. For $e^{tX}, e^{tY} \in G$ commute for all $t$. If $t$ is sufficiently small we can take logs, and deduce that $tX = \log(e^{tX}), tY = \log(e^{tY})$ commute. Hence $X, Y$ commute.*

*The map*

$$\Theta : \mathscr{L}G \to G$$

*under which*

$$X \mapsto e^X$$

*is a homomorphism, since*

$$X + Y \mapsto e^{X+Y} = e^X e^Y.$$

*For any linear group $G$, there exist open subsets $U \ni 0$ in $\mathscr{L}G$, $V \ni I$ in $G$ such that $X \mapsto e^X$ defines a homeomorphism $U \to V$.*

*It follows that $\operatorname{im}\Theta \subset \mathscr{L}G$ is an open subgroup of $G$. Since $G$ is connected, $\operatorname{im}\Theta = G$. Thus*

$$G \cong \mathscr{L}G / \ker\Theta.$$

*Moreover, $\ker\Theta$ is discrete, since $U \cap \ker\Theta = \{0\}$. Thus*

$$G \cong \mathbb{R}^n / K,$$

*where $K$ is a discrete subgroup, and $n = \dim G$.*

Lemma: *A discrete subgroup $K \subset \mathbb{R}^n$ is necessarily $\cong \mathbb{Z}^d$ for some $d \le n$, ie we can find a $\mathbb{Z}$-basis $k_1, \dots, k_d$ for $K$ such that*

$$K = \{n_1 k_1 + \cdots + n_d k_d : n_1, \dots, n_d \in \mathbb{Z}\}.$$

Proof: *Let $k_1$ be one of the closest points to 0 in $K \setminus \{0\}$. Then let $k_2$ be one of the closest points to the subspace $\langle k_1 \rangle$ in $K \setminus \langle k_1 \rangle$, let $k_3$ be one of the closest points to the subspace $\langle k_1, k_2 \rangle$ in $K \setminus \langle k_1, k_2 \rangle$, and so on.*

*Then $k_1, k_2, \dots$ are linearly independent. So the process must end after $d \le n$ steps:*

$$K = \langle k_1, \dots, k_d \rangle.$$

*Now suppose $k \in K$, say*

$$k = \lambda_1 k_1 + \cdots + \lambda_d k_d.$$

*We show that $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$. Let*

$$\lambda_d = r + \epsilon,$$

*where $r \in \mathbb{Z}$ and $|\epsilon| \leq 1/2$. Then*

$$k - r k_d = \lambda_1 k_1 + \cdots + \lambda_{d-1} k_{d-1} + \epsilon k_d$$

*is closer to $\langle k_1, \ldots, k_{d-1} \rangle$ than is $k_d$. Hence $\epsilon = 0$, ie $\lambda_d \in \mathbb{Z}$.*
*Applying the same argument to*

$$k - \lambda_d k_d = \lambda_1 k_1 + \cdots + \lambda_{d-1} k_{d-1},$$

*we deduce that $\lambda_{d-1} \in \mathbb{Z}$; and so successively $\lambda_{d-2}, \ldots, \lambda_1 \in \mathbb{Z}$*
*Thus $k_1, \ldots, k_d$ is a $Z$-basis for $K$. Extend $k_1, \ldots, k_d$ to a basis $k_1, \ldots, k_d, e_1, \ldots, e_{n-d}$ of $\mathbb{R}^n$. Then*

$$\begin{aligned} G \cong \mathbb{R}^n / K &\cong \mathbb{R}/\mathbb{Z} \oplus \cdots \oplus \mathbb{R}/\mathbb{Z} \\ &\cong \mathbb{T}^d \oplus \mathbb{R}^{n-d}. \end{aligned}$$

*Since $G$ is compact, $n - d = 0$, ie*

$$G \cong \mathbb{T}^n.$$